Ranking Median Regression: Learning to Order through Local Consensus

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Outline

1. Ranking Regression
2. Background and Results on Ranking Aggregation
3. Risk Minimization for Ranking (Median) Regression
4. Algorithms - Local Median Methods

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Consider:

- A set of n items: $[\![n]\!] = \{1, \dots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

Ex: $\sigma(2) = 1, \sigma(1) = 2, \dots$
$\mathfrak{S}_n$: set of permutations of $[\![n]\!]$, the symmetric group.

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Related Work

- Has been referred to as **label ranking** in the literature [Tsoumakas et al., 2009], [Vembu and Gärtner, 2010]
- Related to multiclass and multilabel classification
- A lot of applications, e.g : document categorization, meta-learning
  - rank a set of topics relevant for a given document
  - rank a set of algorithms according to their suitability for a new dataset, based on the characteristics of the dataset
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$\Rightarrow$ We develop an approach free of any parametric assumptions (**local learning**) relying on results and framework developped in [Korba et al., 2017] for **ranking aggregation**.

Our Problem

Suppose we observe $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$ i.i.d. copies of the pair $(X,\ \Sigma)$, where

- $X \sim \mu$, where μ is a distribution on some feature space $\mathcal{X}$
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Goal: Learn a predictive ranking rule :

$$\begin{array}{rccc} s & : & \mathcal{X} & \rightarrow & \mathfrak{S}_n \\ & & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Piecewise Constant Ranking Rules

Our approach: build *piecewise constant* ranking rules, i.e:
Ranking rules that are constant on each cell of a partition of $\mathcal{X}$ built from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

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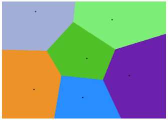
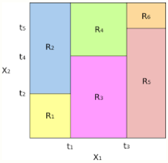
Two methods are investigated:

- k-nearest neighbor (Voronoi partitioning)

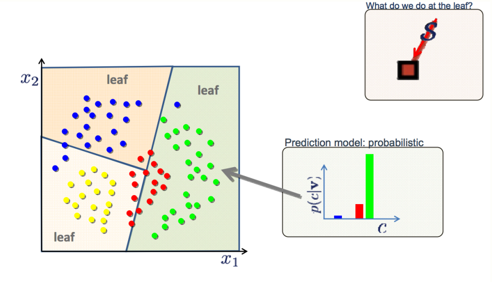


- decision tree (Recursive partitioning)



Compute Local Labels/Medians

For classification, the label of a cell (ex: a leaf) is the **majority** label among the training data which fall in this cell.



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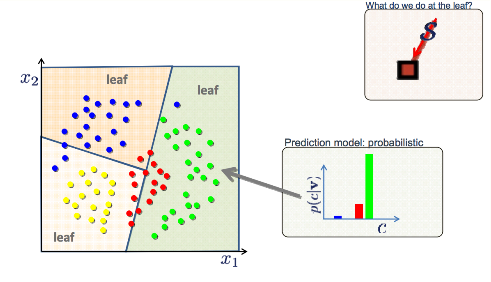

Problem: Our labels are *permutations* σ:

For a cell $\mathcal{C}_k$, if $\sigma_1, \ldots, \sigma_N \in \mathcal{C}_k$, how do we aggregate them into a final label σ^*?

$\Longrightarrow$ Ranking aggregation problem.

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Ranking Aggregation

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \dots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

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Kemeny's rule (1959)

Find the solution of :

$$\sigma^* = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}} \sum_{k=1}^{N} d(\sigma, \sigma_k)$$

where d is the Kendall's tau distance:

$$d_\tau(\sigma, \sigma') = \sum_{i<j} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\},$$

Ex: σ= 1234, σ'= 2413 $\Rightarrow d_\tau(\sigma, \sigma') = 3$ (disagree on (12),(14),(34)).

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Problem: Solving (1) is NP-hard.

Statistical Ranking Aggregation [Korba et al., 2017]

Probabilistic Modeling

$$\mathcal{D}_N = (\Sigma_1, \ldots, \Sigma_N) \qquad \text{with} \qquad \Sigma_k \sim P$$

where P distribution on $\mathfrak{S}_n$.

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Definition

A **Kemeny median** of P is solution of:

$$\sigma^*_P = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}}\, L_P(\sigma), \tag{1}$$

where $L_P(\sigma) = \mathbb{E}_{\Sigma \sim P}[d(\sigma, \Sigma)]$ is **the risk** of σ.

Question: Can we exhibit some conditions on P so that solving (1) is tractable?

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ (probability that item $i \succ j$).

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$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

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Suppose P satisfies **SST and NA**(h) for a given $h > 0$. Then with overwhelming probability $1 - \frac{n(n-1)}{4} e^{-\alpha_h N}$:

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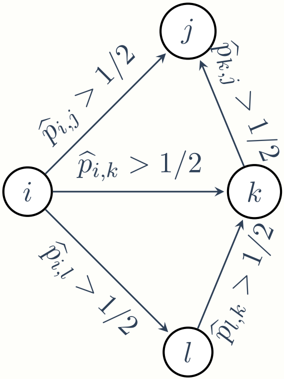
$\widehat{P}$ also verifies **SST**...and the Kemeny median of P is given by the empirical Copeland ranking:

$$\sigma^*_P(i) = 1 + \sum_{j \neq i} \mathbb{I}\{\widehat{p}_{i,j} < \frac{1}{2}\} \quad \text{for } 1 \leq i \leq n$$

Graph of pairwise probabilities



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$\Rightarrow$ sort the i's by increasing input degree

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Performance: Measured by the risk:

$$\mathcal{R}(s) = \mathbb{E}_{X \sim \mu, \Sigma \sim P_X} \left[d_\tau \left(s(X), \Sigma \right) \right]$$

Optimal Elements and Relaxation

$$\mathcal{R}(s) = \mathbb{E}_{X\sim\mu}\left[\mathbb{E}_{\Sigma\sim P_X}\left[d_\tau\left(s(X),\Sigma\right)\right]\right] = \mathbb{E}_{X\sim\mu}\left[L_{P_X}(s(X))\right] \quad (2)$$

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Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \text{argmin}_{\sigma\in\mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

Optimal elements

The predictors s^* minimizing (2) are the ones that maps any point $X \in \mathcal{X}$ to the **conditional** Kemeny median of P_X:

$$s^* = \underset{s\in\mathcal{S}}{\text{argmin}}\, \mathcal{R}(s) \quad \Leftrightarrow \quad s^*(X) = \sigma^*_{P_X}$$

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To minimize the risk (2) approximately:

$$\sigma^*_{P_X} \text{ for any } X \Longrightarrow \sigma^*_{P_\mathcal{C}} \text{ for any } X \in \mathcal{C}$$

where $P_\mathcal{C}(\sigma) = \mathbb{P}[\Sigma = \sigma | X \in \mathcal{C}]$.

$\Rightarrow$ We develop Local consensus methods.

Statistical Framework- ERM

Optimize a statistical version of the theoretical risk based on the training data (X_k, Σ_k)'s:

$$\min_{s \in \mathcal{S}} \widehat{\mathcal{R}}_N(s) = \frac{1}{N} \sum_{k=1}^{N} d_\tau(s(X_k),\ \Sigma_k)$$

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⇒ We consider a subset $\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}$:

- rich enough so that $\inf_{s \in \mathcal{S}_{\mathcal{P}}} \mathcal{R}(s) - \inf_{s \in \mathcal{S}} \mathcal{R}(s)$ is ”small”
- ideally appropriate for greedy optimization.

⇒ $\mathcal{S}_{\mathcal{P}}$= space of piecewise constant ranking rules (”local consensus methods”)

Theorem

Suppose that:

There exists $M < \infty$ such that:

$\forall (x, x') \in \mathcal{X}^2, \;\; \sum_{i<j} |p_{i,j}(x) - p_{i,j}(x')| \leq M \cdot ||x - x'||$.

Then:

$$\mathcal{R}(s^*_{\mathcal{P}}) - \mathcal{R}(s^*) \leq M.\delta_{\mathcal{P}}$$

where $\delta_{\mathcal{P}}$ is the max. diameter of $\mathcal{P}$'s cells.

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Suppose in addition that:

For all $x \in \mathcal{X}$, $P_x \in \mathcal{T}$ and $H = \inf_{x \in \mathcal{X}} \min_{i<j} |p_{i,j}(x) - 1/2| > 0.$

and that $P_{\mathcal{C}} \in \mathcal{T}$ for all $\mathcal{C} \in \mathcal{P}$.

Then,

$$\mathbb{E}\left[d_\tau\left(\sigma^*_{P_X}, s^*_{\mathcal{P}}(X)\right)\right] \leq \sup_{x \in \mathcal{X}} d_\tau\left(\sigma^*_{P_x}, s^*_{\mathcal{P}}(x)\right) \leq (M/H) \cdot \delta_{\mathcal{P}}$$

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Partitioning Methods

Goal: Generate partitions $\mathcal{P}_N$ from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

For $\mathcal{C} \in \mathcal{P}_N$, consider its empirical distribution:

$$\widehat{P}_{\mathcal{C}} = \frac{1}{N_{\mathcal{C}}} \sum_{k : X_k \in \mathcal{C}} \delta_{\Sigma_k}$$

and compute locally itsEmpirical Kemeny median $\widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}$.

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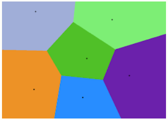
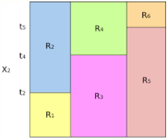


- decision tree (Recursive partitioning)



K-Nearest Neigbors Algorithm

1. Fix $k \in \{1, \ldots, N\}$ and a query point $x \in \mathcal{X}$
2. Sort $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$ by increasing order of the distance to x : $\|X_{(1,N)} - x\| \leq \ldots \leq \|X_{(N,N)} - x\|$
3. Consider next the empirical distribution calculated using the k training points closest to x

$$\widehat{P}(x) = \frac{1}{k} \sum_{l=1}^{k} \delta_{\Sigma_{(l,N)}}$$

and compute the pseudo-empirical Kemeny median, yielding the k-NN prediction at x:

$$s_{k,N}(x) \overset{def}{=} \widetilde{\sigma}^*_{\widehat{P}(x)}.$$

$\Rightarrow$ We recover the classical bound $\mathcal{R}(s_{k,N}) - \mathcal{R}^* = \mathcal{O}(\frac{1}{\sqrt{k}} + \frac{k}{N})$

Decision Tree

Split recursively the feature space $\mathcal{X}$ to minimize some impurity criterion.

Analog to Gini criterion in classification: m classes, f_i proportion of class $i \to I_G(f) = \sum_{i=1}^{m} f_i(1 - f_i)$

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Here, for a cell $\mathcal{C} \in \mathcal{P}_N$:

- Impurity [Alvo and Philip, 2014]:

$$\gamma_{\widehat{P}_{\mathcal{C}}} = \frac{1}{2} \sum_{i<j} \widehat{p}_{i,j}(\mathcal{C}) \left(1 - \widehat{p}_{i,j}(\mathcal{C})\right)$$

  which is tractable and satisfies the double inequality

$$\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}} \leq \min_{\sigma \in \mathfrak{S}_n} L_{\widehat{P}_{\mathcal{C}}}(\sigma) \leq 2\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}}.$$

- Terminal value : Compute the pseudo-empirical median of a cell $\mathcal{C}$:

$$s_{\mathcal{C}}(x) \stackrel{def}{=} \widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}.$$

Conclusion

Interesting challenges:

- Most of the maths from euclidean spaces cannot be applied
- But our insights still hold
- Based on our results on ranking aggregation, we develop a novel approach to ranking regression/label ranking
- Theoretical guarantees (approximation error, rates of convergence)
- We propose two practical algorithms

Openings: How to extend to incomplete rankings (+with ties)?

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In *Preference learning*, pages 45–64. Springer.

In practice: Pseudo-empirical Kemeny Medians

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- If $\widehat{P}$ is SST, compute $\sigma^*_{\widehat{P}}$ with Copeland method based on $\widehat{p}_{i,j}$
- Else, compute $\widetilde{\sigma}^*_{\widehat{P}}$ with empirical Borda count ([Jiang et al., 2011])

$$\widetilde{\sigma}^*_{\widehat{P}}(i) = \frac{1}{N}\sum_{k=1}^{N} \Sigma_k(i) \quad \text{for } 1 \le i \le n$$

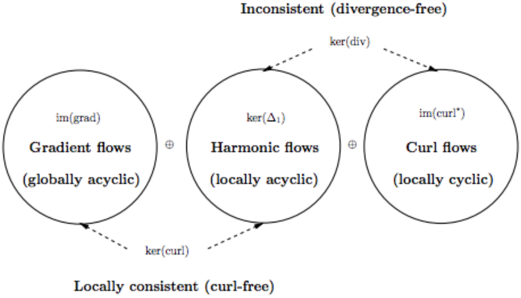

FIGURE 2. Hodge/Helmholtz decomposition of pairwise rankings