Controlling the distance to the Kemeny consensus without computing it

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Outline

Ranking aggregation and Kemeny's rule

State of the art and contribution

Geometric analysis of Kemeny aggregation

Geometric interpretation and proof of the main result

Numerical experiments

Conclusion

Ranking aggregation

Problem:

How to summarize a collection of rankings into one ranking?

Input

- Set of items: $[\![n]\!] := \{1, \ldots, n\}$
- N Rankings of the form : $i_1 \succ i_2 \succ \cdots \succ i_n$

Output

A global order ("consensus") σ^* on the n objects.

Applications

Example 1: Elections

- Let a set of candidates $\{A, B, C, D\}$.
- Each voter gives a full ranking of candidates, for example: $B \succ D \succ A \succ C$
- The set of votes for the election is a **full rankings datasets**.

$\Rightarrow$ How to elect the winner?

Borda-Condorcet debate from 18^{th} century

Jean-Charles de Borda

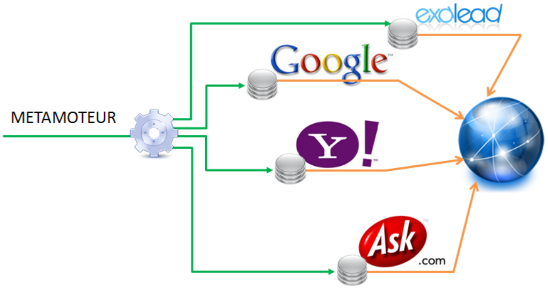
Nicolas de Condorcet

Applications

Example 2: Meta-search engines

For a given query q, a meta-search engine returns the results of several search engines.

$\Rightarrow$ How can we aggregate the ordered lists of all these search engines?

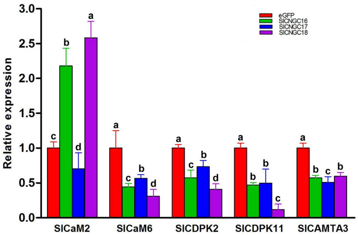


Applications

Exemple 3: Gene expression

- Development of DNA micro-chips enables to measure simultaneous levels of expression for thousands of genes.
- But these measures can vary greatly in scale!
- A possibility is to order genes by their level of expression in each experiment.

$\Rightarrow$ How to agregate the results of all these experiments?



Ranking aggregation

Ranking $i_1 \succ \cdots \succ i_n$ on $[\![n]\!]$ $\Longleftrightarrow$ permutation σ on $[\![n]\!]$ s.t. $\sigma(i_j) = j$.

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What permutation $\sigma^* \in \mathfrak{S}_n$ best represents a given a collection of permutations $(\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$?

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Definition (*Consensus ranking (Kemeny, 1959)*)

A permutation $\sigma^* \in \mathfrak{S}_n$ is a best representative of the collection $(\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$ with respect to a metric d on $\mathfrak{S}_n$ if it is a solution of :

$$min_{\sigma \in \mathfrak{S}_n} \sum_{t=1}^{N} d(\sigma, \sigma_t).$$

Kemeny's rule

Definition (*Kendall's tau distance*)

The Kendalls tau distance between two permutations is equal to the number of their pairwise disagreements:

$$d(\sigma, \pi) = \sum_{\{i,j\} \subset [\![n]\!]} \mathbb{I}\{\sigma \textit{ and } \pi \textit{ disagree on } \{i, j\}\}$$

Example

$\sigma = 123$ $(1 \succ 2 \succ 3)$
$\pi = 231$ $(2 \succ 3 \succ 1)$
$\rightarrow$ number of desagreements = on 2 pairs (12,13).

Kemeny aggregation

Definition (*Kemeny's rule*)

Compute the exact **Kemeny consensus(es)** *for the Kendall's tau distance.*

$$min_{\sigma \in \mathfrak{S}_n} \sum_{t=1}^{N} d(\sigma, \sigma_t) \quad (1)$$

where d is the **Kendall's tau distance**.

Kemeny's rule

- Social choice justification: Satisfies many voting properties, such as the Condorcet criterion: if an alternative is preferred to all others in pairwise comparisons then it is the winner [Young and Levenglick, 1978]
- Statistical justification: Outputs the maximum likelihood estimator under the Mallows model [Young, 1988]
- Main drawback: It is NP-hard in the number of items n [Bartholdi et al., 1989] even for $N = 4$ votes [Dwork et al., 2001].

Outline

Contribution

Previous contributions

- General guarantees for approximation procedures ([Coppersmith 2006], [Ailon 2008])
- Bounds on the approximation cost, computed from the dataset ([Conitzer 2006], [Sibony 2014])
- Conditions for the exact Kemeny aggregation to become tractable ([Betzler 2008])

Contribution

Setting

- Set of items $\llbracket n \rrbracket := \{1, \dots, n\}$
- A rankings dataset $\mathcal{D}_N = (\sigma_1, \dots, \sigma_N) \in \mathfrak{S}_n^N$
- Let $\sigma^* \in \mathcal{K}_N$ a Kemeny consensus
- Let $\sigma \in \mathfrak{S}_n$ a permutation, typically output by a computationally efficient aggregation procedure on $\mathcal{D}_N$.

Our contribution

We give an upper bound on $d(\sigma, \sigma^*)$ by using only tractable quantities.

Remark: The Kendall's distance takes values between 0 and $\frac{n \times (n-1)}{2}$ (the maximal number of disagreements is the number of pairs).

Outline

Kemeny embedding

The Kemeny embedding is the mapping $\phi : \mathfrak{S}_n \to \mathbb{R}^{\binom{n}{2}}$ defined by:

$$\phi : \sigma \mapsto \begin{pmatrix} \vdots \\ sign(\sigma(i) - \sigma(j)) \\ \vdots \end{pmatrix}_{1 \leq i < j \leq n}$$

where $sign(x) = 1$ if $x \geq 0$ and -1 otherwise.

Example

$$123 \mapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{array}{l} \to \text{pair } 12 \\ \to \text{pair } 13 \\ \to \text{pair } 23 \end{array}, 132 \mapsto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{array}{l} \to \text{pair } 12 \\ \to \text{pair } 13 \\ \to \text{pair } 23 \end{array}$$

Kemeny aggregation in $\mathbb{R}^{\binom{n}{2}}$

Definition (*Mean embedding*)

For $D_N = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$, we define the **barycenter***:*

$$\phi(\mathcal{D}_N) := \frac{1}{N} \sum_{t=1}^{N} \phi(\sigma_t).$$

Kemeny aggregation in $\mathbb{R}^{\binom{n}{2}}$

Proposition (*Barthelemy & Monjardet (1981)*)

For all $\sigma, \sigma' \in \mathfrak{S}_n$,

$$\|\phi(\sigma)\| = \sqrt{\frac{n(n-1)}{2}} \quad \textit{and} \quad \|\phi(\sigma) - \phi(\sigma')\|^2 = 4d(\sigma, \sigma'),$$

and for any dataset $\mathcal{D}_N = (\sigma_1, \dots \sigma_N) \in \mathfrak{S}_n^N$, *Kemeny's rule (1) :*

$$min_{\sigma \in \mathfrak{S}_n} \sum_{t=1}^{N} d(\sigma, \sigma_t)$$

is **equivalent** *to the minimization problem*

$$\min_{\sigma \in \mathfrak{S}_n} \|\phi(\sigma) - \phi(\mathcal{D}_N)\|^2 \tag{2}$$

Illustration

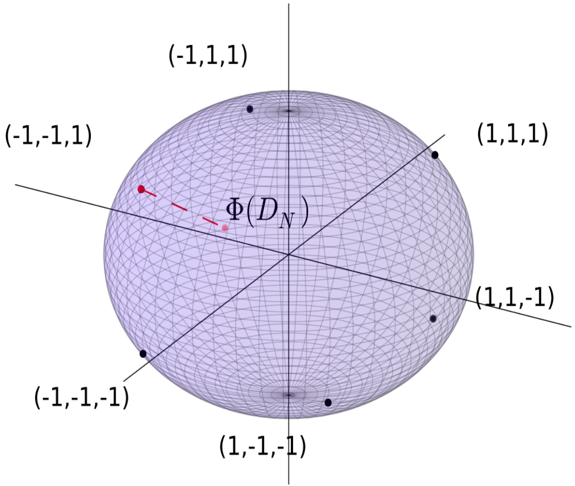

Figure: Kemeny aggregation for $n = 3$.

Kemeny aggregation in $\mathbb{R}^{\binom{n}{2}}$

Kemeny aggregation naturally decomposes in two steps:

1. Compute the barycenter $\phi(\mathcal{D}_N) \in \mathbb{R}^{\binom{n}{2}}$ (complexity $O(Nn^2)$)
2. Find the consensus σ^* solution of problem (2)

Idea: $\Rightarrow \phi(\mathcal{D}_N)$ contains useful information.

Main result

For $\sigma \in \mathfrak{S}_n$, we define the **angle** $\theta_{\mathbf{N}}(\sigma)$ **between** $\phi(\sigma)$ **and** $\phi(\mathcal{D}_{\mathbf{N}})$ by:

$$\cos(\theta_N(\sigma)) = \frac{\langle \phi(\sigma), \phi(\mathcal{D}_N) \rangle}{\|\phi(\sigma)\| \|\phi(\mathcal{D}_N)\|},$$

with $0 \leq \theta_N(\sigma) \leq \pi$.

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Theorem

Let $\mathcal{D}_N \in \mathfrak{S}_n^N$ be a dataset, $\mathcal{K}_N$ the set of Kemeny consensuses and $\sigma \in \mathfrak{S}_n$ a permutation. For any $k \in \{0, \ldots, \binom{n}{2} - 1\}$, one has the following implication:

$$\cos(\theta_N(\sigma)) > \sqrt{1 - \frac{k+1}{\binom{n}{2}}} \quad \Rightarrow \quad \max_{\sigma^* \in \mathcal{K}_N} d(\sigma, \sigma^*) \leq k.$$

Upper bound and application on the sushi dataset

We define:

$$k_{min}(\sigma; \mathcal{D}_N) = \left\lfloor \binom{n}{2} \sin^2(\theta_N(\sigma)) \right\rfloor . \tag{3}$$

the minimal $k \in \{0, \dots, \binom{n}{2} - 1\}$ verifying the theorem condition.

Voting rule	$\cos(\theta_N(\sigma))$	$k_{min}(\sigma)$
Borda	0.82	14
Copeland	0.82	14
QuickSort	0.82	14
Plackett-Luce	0.80	15
2-approval	0.74	20
1-approval	0.71	22
Pick-a-Perm	0.40	37
Pick-a-Random	0.28	41

Outline

Extended cost function

Kemeny aggregation:

$$\min_{\sigma \in \mathfrak{S}_n} C'_N(\sigma) = \|\phi(\sigma) - \phi(\mathcal{D}_N)\|^2.$$

Relaxed problem:

$$\min_{x \in \mathbb{S}} \mathcal{C}_N(x) := \|x - \phi(\mathcal{D}_N)\|^2.$$

Illustration

For any $x \in \mathbb{S}$, by denoting R the radius of $\mathbb{S}$, one has:

$$\mathcal{C}_N(x) = R^2 + \|\phi(\mathcal{D}_N)\|^2 - 2R\|\phi(\mathcal{D}_N)\| \cos(\theta_N(x)).$$

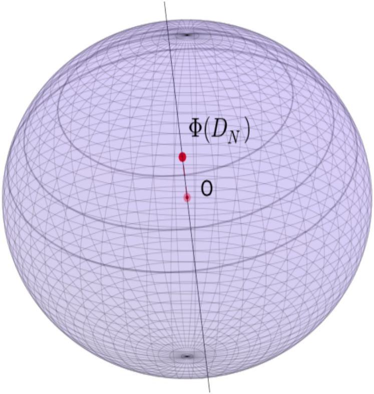

Figure: Level sets of $\mathcal{C}_N$

Lemmas

Lemma (1)

A Kemeny consensus of a dataset $\mathcal{D}_N$ is a permutation σ^* s.t:

$$\theta_N(\sigma^*) \leq \theta_N(\sigma) \qquad \text{for all } \sigma \in \mathfrak{S}_n.$$

Lemma (2)

For $x \in \mathbb{S}$ and $r \geq 0$, one has:

$$\cos(\theta_N(x)) > \sqrt{1 - \frac{r^2}{4R^2}} \Rightarrow \min_{x' \in \mathbb{S} \setminus \mathcal{B}(x,r)} \theta_N(x') > \theta_N(x).$$

Illustration

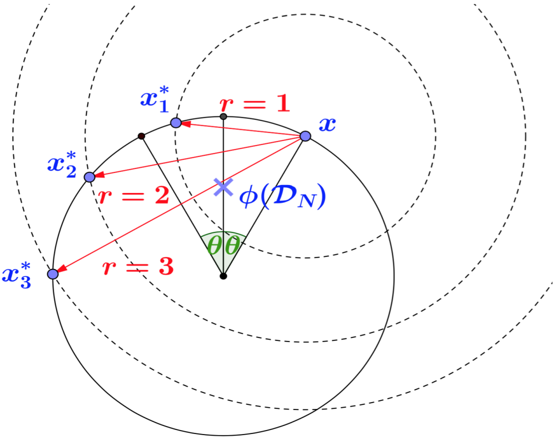

Figure: Illustration of Lemma 2 with r taking integer values (representing possible Kendall's tau distance). Here minimum r satisfying the condition is 2.

Embedding of a ball

Lemma (3)

For $\sigma \in \mathfrak{S}_n$ and $k \in \{0, \ldots, \binom{n}{2}\}$,

$$\phi\left(\mathfrak{S}_n \setminus B(\sigma, k)\right) \quad \subset \quad \mathbb{S} \setminus \mathcal{B}(\phi(\sigma), 2\sqrt{k+1})$$

Outline

Tightness of the bound

We denote by:

- n the number of items
- $\mathcal{D}_N \in \mathfrak{S}_n^N$ any dataset
- σ^* the Kemeny consensus
- r any voting rule, and by σ the consensuses of $\mathcal{D}_N$ given by r

We know that:

$$d(\sigma, \sigma^*) \leq k_{min} .$$

The tightness of the bound is the difference between our upper bound and the real distance:

$$s\left(r, \mathcal{D}_N, n\right) := k_{min} - d(\sigma, \sigma^*) .$$

Results

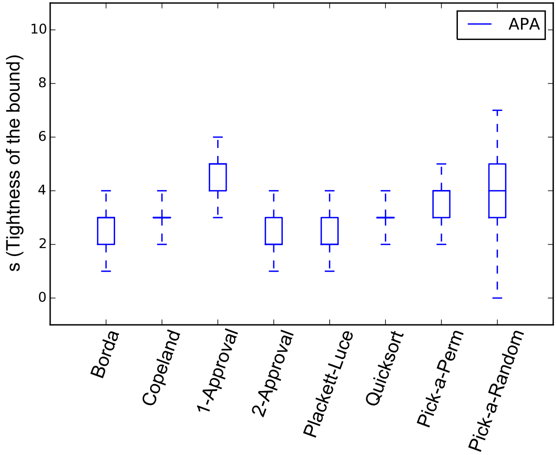

Figure: Boxplot of $s\left(r, \mathcal{D}_N, n\right)$ over sampling collections of datasets shows the effect from different voting rules r with 500 bootstrapped pseudo-samples of the APA dataset $(n = 5, N = 5738)$.

Predictability of the method

- When n grows, the exact Kemeny consensus σ^* quickly becomes computationally impermissible.

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- Once we have an approximate ranking σ and k_{min} is identified via our method, the search scope for the exact Kemeny consensuses can be **narrowed down** to those permutations within a distance of k_{min} to σ.

- The total number of such permutations in $\mathfrak{S}_n$ is upper bounded by $\binom{n+k_{min}-1}{k_{min}} << |\mathfrak{S}_n| = n!$ [Wang 2013].

Results

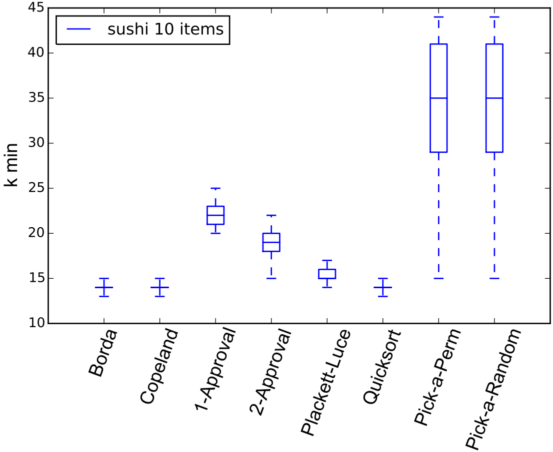

Figure: Boxplot of k_{min} over 500 bootstrapped pseudo-samples of the sushi dataset ($n = 10$, $N = 5000$).

Outline

Conclusion

- We have established a theoretical result that allows to control the Kendall's tau distance between a permutation and the Kemeny consensuses of any dataset.
- This provides a simple and general method to predict, for any ranking aggregation procedure, how close the outcome on a dataset is from the Kemeny consensuses.

Future directions

- The geometric properties of the Kemeny embedding are rich and could lead to many more results.
- We can imagine ranking aggregation procedures using a smaller scope for Kemeny consensuses.
- Possible extensions to incomplete rankings.

Thank you