Ranking Median Regression: Learning to Order through Local Consensus

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Outline

1. Introduction to Ranking Data
2. Ranking Regression
3. Background on Ranking Aggregation/Medians
4. Risk Minimization for Ranking (Median) Regression
5. Algorithms - Local Median Methods
6. Ongoing work - Structured prediction methods
7. Conclusion

Outline

Ranking Data

Set of items $[\![n]\!] := \{1, \ldots, n\}$

Definition (Ranking)

A ranking is a strict partial order $\prec$ over $[\![n]\!]$, *i.e.* a binary relation satisfying the following properties:

Irreflexivity For all $i \in [\![n]\!]$, $i \not\prec i$

Transitivity **For all i, j, k $\in [\![n]\!]$, if i $\prec$ j and j $\prec$ k then i $\prec$ k**

Asymmetry For all $i, j \in [\![n]\!]$, if $i \prec j$ then $j \not\prec i$

Common types of rankings

Set of items $[\![n]\!] := \{1, \ldots, n\}$

- **Full ranking.** All the items are ranked, without ties

$$a_1 \succ a_2 \succ \cdots \succ a_n$$

- **Partial ranking.** All the items are ranked, with ties (”buckets”)

$$a_{1,1}, \ldots, a_{1,n_1} \succ \cdots \succ a_{r,1}, \ldots, a_{r,n_r} \quad \text{with} \quad \sum_{i=1}^{r} n_i = n$$

  $\Rightarrow$ includes **Top-k ranking**: $a_1, \ldots, a_k \succ$ the rest

- **Incomplete ranking.** Only a subset of items are ranked, without ties

$$a_1 \succ \cdots \succ a_k \quad \text{with} \quad k < n$$

  $\Rightarrow$ includes **Pairwise comparison**: $a_1 \succ a_2$

Ranking data arise in a lot of applications

Historical applications

- **Elections**: $[\![n]\!]$= a set of candidates
  $\rightarrow$ A voter ranks a set of candidates
- **Surveys**: $[\![n]\!]$= political goals
  $\rightarrow$ A citizen ranks according to its priorities
- **Competitions**: $[\![n]\!]$= a set of players
  $\rightarrow$ Results of a race

Modern applications

- **E-commerce**: $[\![n]\!]$= items of a catalog
  $\rightarrow$ A user expresses its preferences (see ”implicit feedback”)
- **Search engines**: $[\![n]\!]$= web-pages
  $\rightarrow$ A search engine ranks by relevance for a given query
- **Biology**: $[\![n]\!]$= genes [Jiao and Vert, 2015], brain regions [Gunasekar et al., 2016]
  $\rightarrow$ Rank the items by level of expression/associations

Detailed example: analysis of full rankings

Consider:

- A set of n items: $[\![n]\!] = \{1, \dots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A full ranking: $a_1 \succ a_2 \succ \cdots \succ a_n$ (Ex: $2 \succ 1 \succ 3 \succ 4$)
- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

  Ex: $\sigma(2) = 1, \sigma(1) = 2, \cdots \Rightarrow \sigma = 2134$

- $\mathfrak{S}_n$: set of permutations of $[\![n]\!]$, the symmetric group.
  Ex: $\mathfrak{S}_4 = 1234, 1324, 1423, \dots, 4321$

Probabilistic Modeling. The dataset is a collection of random permutations drawn IID from a probability distribution P over $\mathfrak{S}_n$:

$$\mathcal{D}_N = (\Sigma_1, \dots, \Sigma_N) \in \mathfrak{S}_n^N \qquad \text{with} \qquad \Sigma_i \sim P$$

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- A random permutation $\Sigma \in \mathfrak{S}_n$ can be seen as a random vector $(\Sigma(1), \ldots, \Sigma(n)) \in \mathbb{R}^n$... but

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- Apply a method from p.d.f. estimation (e.g. kernel density estimation)... but
  No canonical ordering of the rankings!

Main approaches

“Parametric” approach

- Fit a predefined generative model on the data
- Analyze the data through that model

“Nonparametric” approach

- Choose a structure on $\mathfrak{S}_n$
- Analyze the data with respect to that structure

Parametric Approach - Example of Models

- **Mallows model** [Mallows, 1957]
  Parameterized by a central ranking $\sigma_0 \in \mathfrak{S}_n$ and a dispersion parameter $\gamma \in \mathbb{R}^+$

$$P(\sigma) = Ce^{-\gamma d(\sigma_0, \sigma)} \qquad \text{with } d \text{ a distance on } \mathfrak{S}_n.$$

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- **Plackett-Luce model** [Luce, 1959], [Plackett, 1975]
  Each item i is parameterized by w_i with $w_i \in \mathbb{R}^+$:

  $$P(\sigma) = \prod_{i=1}^{n} \frac{w_{\sigma_i}}{\sum_{j=i}^{n} w_{\sigma_j}}$$

  Ex: $2 \succ 1 \succ 3 = \frac{w_2}{w_1+w_2+w_3} \frac{w_1}{w_1+w_3}$

Nonparametric approaches - Examples 1

- Harmonic analysis
  - Fourier analysis [Clémençon et al., 2011], [Kondor and Barbosa, 2010]

$$\hat{h}_\lambda = \sum_{\sigma \in \mathfrak{S}_n} h(\sigma)\rho_\lambda(\sigma) \text{ où } \rho_\lambda(\sigma) \in \mathbb{C}^{d_\lambda \times d_\lambda} \text{ for all } \lambda \vdash n.$$

  - Multiresolution analysis for incomplete rankings [Sibony et al., 2015]

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  - Multiresolution analysis for incomplete rankings [Sibony et al., 2015]
- Embeddings of permutations
  - Permutation matrices [Plis et al., 2011]

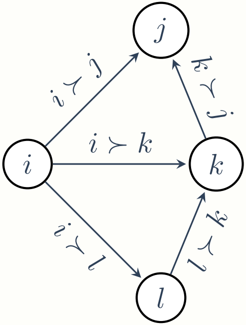
$$\mathfrak{S}_n \to \mathbb{R}^{n \times n}, \quad \sigma \mapsto P_\sigma \quad \text{with } P_\sigma(i,j) = \mathbb{I}\{\sigma(i) = j\}$$

  - Kemeny embedding [Jiao et al., 2016]

$$\mathfrak{S}_n \to \mathbb{R}^{n(n-1)/2}, \quad \sigma \mapsto \phi_\sigma \quad \text{with } \phi_\sigma = \begin{pmatrix} \vdots \\ sign(\sigma(i) - \sigma(j)) \\ \vdots \end{pmatrix}_{i<j}$$

Nonparametric approaches - Examples 2

Modeling of pairwise comparisons as a graph:



- HodgeRank exploits the topology of the graph [Jiang et al., 2011]
- Approximation of pairwise comparison matrices [Shah and Wainwright, 2015]

Some ranking problems

Perform some task on a dataset of N rankings $\mathcal{D}_N = (\Sigma_1, \dots, \Sigma_N)$.

Examples

- **Top-1 recovery:** Find the "most preferred" item in $\mathcal{D}_N$
  e.g. Output of an election
- **Aggregation:** Find a full ranking that "best summarizes" $\mathcal{D}_N$
  e.g. Ranking of a competition
- **Clustering:** Split $\mathcal{D}_N$ into clusters
  e.g. Segment customers based on their answers to a survey
- **Prediction:** Predict a ranking given some information
  e.g. In a recommendation setting

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Example: n=4 fruits

 $\rightarrow$  $\succ$  $\succ$  $\succ$ 

Related Work

- Has been referred to as **label ranking** in the literature [Tsoumakas et al., 2009], [Vembu and Gärtner, 2010]
- Can be seen as an extension of multiclass and multilabel classification
- A lot of applications, e.g : document categorization, meta-learning
  - rank a set of topics relevant for a given document
  - rank a set of algorithms according to their suitability for a new dataset, based on the characteristics of the dataset
- A lot of approaches rely on parametric modelling [Cheng and Hüllermeier, 2009], [Cheng et al., 2009], [Cheng et al., 2010]

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$\Rightarrow$ We develop an approach free of any parametric assumptions (**local learning**) relying on results and framework developped in [Korba et al., 2017] for **ranking aggregation**.

Problem and Setting

Suppose we observe $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$ i.i.d. copies of the pair $(X,\ \Sigma)$, where

- $X \sim \mu$, where μ is a distribution on some feature space $\mathcal{X}$
- $\Sigma \sim P_X$, where P_X is the conditional probability distribution (on $\mathfrak{S}_n$): $P_X(\sigma) = \mathbb{P}[\Sigma = \sigma | X]$

Ex: Users i with characteristics X_i order items by preference resulting in Σ_i.

Goal: Learn a predictive ranking rule :

$$\begin{array}{cccc} s & : & \mathcal{X} & \rightarrow & \mathfrak{S}_n \\ & & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Objective

Performance: Measured by the risk:

$$\mathcal{R}(s) = \mathbb{E}_{X \sim \mu, \Sigma \sim P_X} \left[d_\tau \left(s(X), \Sigma \right) \right]$$

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where d_τ is the Kendall's tau distance, i.e. for $\sigma, \sigma' \in \mathfrak{S}_n$:

$$d_\tau(\sigma, \sigma') = \sum_{1 \leq i < j \leq n} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\},$$

Ex: σ= 1234, σ'= 2413 $\Rightarrow d_\tau(\sigma, \sigma') = 3$ (disagree on (12),(14),(34)).

Piecewise Constant Ranking Rules

Our approach: build *piecewise constant* ranking rules, i.e:
Ranking rules that are constant on each cell of a partition of $\mathcal{X}$ built from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

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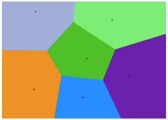
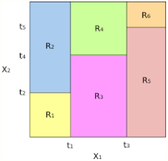
Two methods are investigated:

- k-nearest neighbor (Voronoi partitioning)

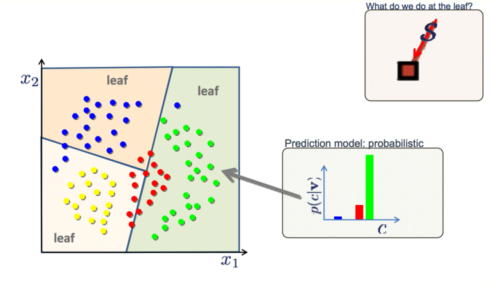


- decision tree (Recursive partitioning)



Compute Local Labels/Medians

For classification, the label of a cell (ex: a leaf) is the **majority** label among the training data which fall in this cell.



4 classes: green, red, blue, yellow $\rightarrow$ green will be the label for the right cell.

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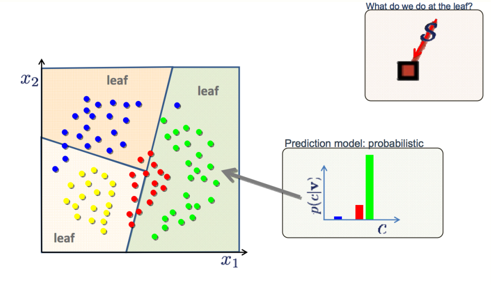

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Problem: Our labels are *permutations* σ:

For a cell $\mathcal{C}$, if $\Sigma_1, \ldots, \Sigma_N \in \mathcal{C}$, how do we aggregate them into a final label σ^*?

$\Longrightarrow$ Ranking aggregation problem.

Outline

Ranking Aggregation - Methods

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \dots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

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Kemeny's rule (1959) - Optimization pb

$$\text{Solve} \quad \sigma^* = \operatorname*{argmin}_{\sigma \in \mathfrak{S}_n} \sum_{k=1}^{N} d(\sigma, \sigma_k)$$

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Problem: NP-hard.

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Copeland method - Scoring method

Sort the items i according to their Copeland score s_C:

$$s_C(i) = \frac{1}{N} \sum_{k=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{I}[\sigma_k(i) < \sigma_k(j)]$$

which counts the number of pairwise victories of item i over the other items $j \neq i \Rightarrow \mathcal{O}(n^2 N)$ complexity.

Statistical Ranking Aggregation [Korba et al., 2017]

Probabilistic Modeling

$$\mathcal{D}_N = (\Sigma_1, \ldots, \Sigma_N) \qquad \text{with} \qquad \Sigma_k \sim P$$

where P distribution on $\mathfrak{S}_n$.

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Definition

A **Kemeny median** of P is solution of:

$$\sigma_P^* = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}}\, L_P(\sigma), \tag{1}$$

where $L_P(\sigma) = \mathbb{E}_{\Sigma \sim P}[d(\sigma, \Sigma)]$ is **the risk** of σ.

Question: Can we exhibit some conditions on P so that solving (1) is tractable?

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

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Strict Stochastic Transitivity (**SST**): $(p_{i,j} \neq 1/2 \ \forall i, j)$

$$p_{i,j} > 1/2 \text{ and } p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$$

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Low-Noise/**NA**(h) for $h > 0$ ([Audibert and Tsybakov, 2007]):

$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

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Our result

Suppose P satisfies **SST and NA**(h) for a given $h > 0$. Then with overwhelming probability $1 - \frac{n(n-1)}{4} e^{-\alpha_h N}$:

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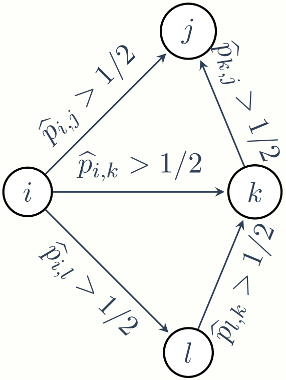
$\widehat{P}$ also verifies **SST**...and the Kemeny median of P is given by the empirical Copeland ranking:

$$\sigma^*_P(i) = 1 + \sum_{j \neq i} \mathbb{I}\{\widehat{p}_{i,j} < \frac{1}{2}\} \quad \text{for } 1 \leq i \leq n$$

Graph of pairwise probabilities



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$\Rightarrow$ sort the i's by increasing input degree

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Goal: Learn a predictive ranking rule :

$$\begin{array}{cccc} s & : & \mathcal{X} & \to & \mathfrak{S}_n \\ & & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Performance: Measured by the risk:

$$\begin{aligned} \mathcal{R}(s) &= \mathbb{E}_{X \sim \mu, \Sigma \sim P_X}\left[d_\tau\left(s(X), \Sigma\right)\right] \\ &= \mathbb{E}_{X \sim \mu}\left[\mathbb{E}_{\Sigma \sim P_X}\left[d_\tau\left(s(X), \Sigma\right)\right]\right] \\ &= \mathbb{E}_{X \sim \mu}\left[L_{P_X}(s(X))\right] \end{aligned}$$

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$\Rightarrow$ Ranking regression is an extension of ranking aggregation.

Optimal Elements and Relaxation

Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \operatorname{argmin}_{\sigma \in \mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

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Optimal elements

The predictors s^* minimizing $\mathcal{R}(s)$ are the ones that maps any point $X \in \mathcal{X}$ to the **conditional** Kemeny median of P_X:

$$s^* = \underset{s \in \mathcal{S}}{\text{argmin}}\, \mathcal{R}(s) \quad \Leftrightarrow \quad s^*(X) = \sigma^*_{P_X}$$

Optimal Elements and Relaxation

Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \text{argmin}_{\sigma \in \mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

Optimal elements

The predictors s^* minimizing $\mathcal{R}(s)$ are the ones that maps any point $X \in \mathcal{X}$ to the **conditional** Kemeny median of P_X:

$$s^* = \underset{s \in \mathcal{S}}{\text{argmin}}\, \mathcal{R}(s) \quad \Leftrightarrow \quad s^*(X) = \sigma^*_{P_X}$$

To minimize the risk $\mathcal{R}(s)$ approximately:

$$\sigma^*_{P_X} \text{ for any } X \Longrightarrow \sigma^*_{P_{\mathcal{C}}} \text{ for any } X \in \mathcal{C}$$

where $P_{\mathcal{C}}(\sigma) = \mathbb{P}[\Sigma = \sigma | X \in \mathcal{C}]$.

$\Rightarrow$ We develop Local consensus methods.

Statistical Framework- ERM

Optimize a statistical version of the theoretical risk based on the training data (X_k, Σ_k)'s:

$$\min_{s \in \mathcal{S}} \widehat{\mathcal{R}}_N(s) = \frac{1}{N} \sum_{k=1}^{N} d_\tau(s(X_k),\ \Sigma_k)$$

where $\mathcal{S}$ is the set of measurable mappings.

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where $\mathcal{S}$ is the set of measurable mappings.

$\Rightarrow$ We consider a subset $\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}$:

- rich enough so that $\inf_{s \in \mathcal{S}_{\mathcal{P}}} \mathcal{R}(s) - \inf_{s \in \mathcal{S}} \mathcal{R}(s)$ is ”small”
- ideally appropriate for greedy optimization.

$\Rightarrow$ $\mathcal{S}_{\mathcal{P}}$= space of piecewise constant ranking rules

Our results

Rates of convergence

- classical rates $\mathcal{O}(1/\sqrt{N})$ for ERM.
- fast rates $\mathcal{O}(1/N)$ under a "uniform" **NA**(h).

Approximation Error

Suppose that:

There exists $M < \infty$ such that:

$\forall (x, x') \in \mathcal{X}^2, \;\; \sum_{i<j} |p_{i,j}(x) - p_{i,j}(x')| \leq M \cdot ||x - x'||$.

Then:

$$\mathcal{R}(s^*_{\mathcal{P}}) - \mathcal{R}(s^*) \leq M.\delta_{\mathcal{P}}$$

where $\delta_{\mathcal{P}}$ is the max. diameter of $\mathcal{P}$'s cells.

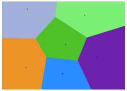
Outline

Partitioning Methods

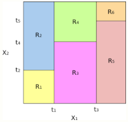
Goal: Generate partitions $\mathcal{P}_N$ from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

Two methods are investigated:

- k-nearest neighbor (Voronoi partitioning)



- decision tree (Recursive partitioning)



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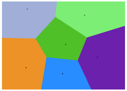

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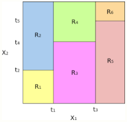

For $\mathcal{C} \in \mathcal{P}_N$, consider its empirical distribution:

$$\widehat{P}_{\mathcal{C}} = \frac{1}{N_{\mathcal{C}}} \sum_{k: X_k \in \mathcal{C}} \delta_{\Sigma_k}$$

Final Labels in Practice

- If $\widehat{P}_{\mathcal{C}}$ is SST, compute $\sigma^*_{\widehat{P}}$ with Copeland method based on the $\widehat{p}_{i,j}$'s

Final Labels in Practice

- If $\widehat{P}_{\mathcal{C}}$ is SST, compute $\sigma^*_{\widehat{P}}$ with Copeland method based on the $\widehat{p}_{i,j}$'s
- Else, compute $\widetilde{\sigma}^*_{\widehat{P}}$ with empirical Borda count ([Jiang et al., 2011])

$$\widetilde{\sigma}^*_{\widehat{P}}(i) = \frac{1}{N}\sum_{k=1}^{N}\Sigma_k(i) \quad \text{for } 1 \leq i \leq n$$

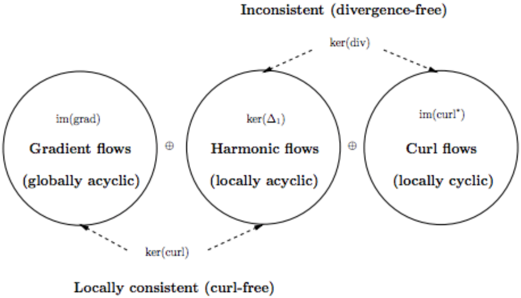

FIGURE 2. Hodge/Helmholtz decomposition of pairwise rankings

K-Nearest Neigbors Algorithm

1. Fix $k \in \{1, \ldots, N\}$ and a query point $x \in \mathcal{X}$
2. Sort $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$ by increasing order of the distance to x : $\|X_{(1,N)} - x\| \leq \ldots \leq \|X_{(N,N)} - x\|$
3. Consider next the empirical distribution calculated using the k training points closest to x

$$\widehat{P}(x) = \frac{1}{k} \sum_{l=1}^{k} \delta_{\Sigma_{(l,N)}}$$

and compute the pseudo-empirical Kemeny median, yielding the k-NN prediction at x:

$$s_{k,N}(x) \overset{def}{=} \widetilde{\sigma}^*_{\widehat{P}(x)}.$$

$\Rightarrow$ We recover the classical bound $\mathcal{R}(s_{k,N}) - \mathcal{R}^* = \mathcal{O}(\frac{1}{\sqrt{k}} + \frac{k}{N})$

Decision Tree

Split recursively the feature space $\mathcal{X}$ to minimize some impurity criterion.

Analog to Gini criterion in multiclassification: m classes, f_i proportion of class $i \rightarrow I_G(\mathcal{C}) = \sum_{i=1}^{m} f_i(\mathcal{C})(1 - f_i(\mathcal{C}))$

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Here, for a cell $\mathcal{C} \in \mathcal{P}_N$:

- Impurity [Alvo and Philip, 2014]:

$$\gamma_{\widehat{P}_{\mathcal{C}}} = \frac{1}{2} \sum_{1 \leq i < j \leq n} \widehat{p}_{i,j}(\mathcal{C}) \left(1 - \widehat{p}_{i,j}(\mathcal{C})\right)$$

  (ordering n elements can be seen as $\binom{n}{2}$ classification tasks) which is tractable and satisfies the double inequality

$$\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}} \leq \min_{\sigma \in \mathfrak{S}_n} L_{\widehat{P}_{\mathcal{C}}}(\sigma) \leq 2\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}}.$$

- Terminal value : Compute the pseudo-empirical median of a cell $\mathcal{C}$:

$$s_{\mathcal{C}}(x) \stackrel{def}{=} \widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}.$$

Simulated Data

- We generate two explanatory variables, varying their nature (numerical, categorical) $\Rightarrow$ Setting 1/2/3
- We generate a partition of the feature space
- On each cell of the partition, a dataset of full rankings is generated, varying the distribution (constant, Mallows with $\neq$ dispersion): $D_0/D_1/D_2$

D_i	Setting 1			Setting 2			Setting 3		
	n=3	n=5	n=8	n=3	n=5	n=8	n=3	n=5	n=8
D_0	0.0698* 0.0473** (0.578)	0.1290* 0.136** (1.147)	0.2670* 0.324** (2.347)	0.0173* 0.0568** (0.596)	0.0405* 0.145** (1.475)	0.110* 0.2695** (3.223)	0.0112* 0.099** (0.5012)	0.0372* 0.1331** (1.104)	0.0862* 0.2188** (2.332)
D_1	0.3475 * 0.307** (0.719)	0.569* 0.529** (1.349)	0.9405 * 0.921** (2.606)	0.306* 0.308** (0.727)	0.494* 0.536** (1.634)	0.784* 0.862** (3.424)	0.289* 0.3374** (0.5254)	0.457* 0.5714** (1.138)	0.668* 0.8544** (2.287)
D_2	0.8656* 0.7228** (0.981)	1.522* 1.322** (1.865)	2.503* 2.226** (3.443)	0.8305 * 0.723** (1.014)	1.447 * 1.3305** (2.0945)	2.359* 2.163** (4.086)	0.8105* 0.7312** (0.8504)	1.437* 1.3237** (1.709)	2.189* 2.252** (3.005)

Table: Empirical risk averaged on 50 trials on simulated data.

(): Clustering +PL, *: K-NN, **: Decision Tree

Outline

Introduction to Ranking Data

Ranking Regression

Background on Ranking Aggregation/Medians

Risk Minimization for Ranking (Median) Regression

Algorithms - Local Median Methods

Ongoing work - Structured prediction methods

Conclusion

Structured prediction approach

Goal: Learn a predictive ranking rule : $s : \mathcal{X} \to \mathfrak{S}_n$
The ranking regression/label ranking problem is then defined as:

$$\min_{s:\mathcal{X}\to\mathfrak{S}_n} \mathcal{R}(s), \quad \text{with } \mathcal{R}(s) = \mathbb{E}_{X \sim \mu, \Sigma \sim P_X} \left[\Delta\left(s(X), \Sigma\right)\right]$$

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Consider a family of loss functions based on some ranking embedding $\phi : \mathfrak{S}_n \to \mathcal{F}$ that maps the permutations $\sigma \in \mathfrak{S}_n$ into a Hilbert space $\mathcal{F}$:

$$\Delta(\sigma, \sigma') = \|\phi(\sigma) - \phi(\sigma')\|_{\mathcal{F}}^2.$$

Motivation:

- Kendall's tau and Hamming distances can be written with Kemeny and Permutation matrices embeddings respectively

Structured prediction approach

$$\text{min}_{s\,:\,\mathcal{X}\to\mathfrak{S}_n}\mathcal{R}(s), \quad \text{with } \mathcal{R}(s) = \mathbb{E}_{X\sim\mu,\Sigma\sim P_X}\left[\Delta\left(s(X),\Sigma\right)\right] \quad (2)$$

and

$$\Delta(\sigma,\sigma') = \|\phi(\sigma)-\phi(\sigma')\|^2_{\mathcal{F}}.$$

We can approach structured prediction (see [Ciliberto et al., 2016, Brouard et al., 2016]) in two steps:

- **Step 1 - Surrogate problem**: Solve an empirical version of (2) by replacing Δ with:

$$L(g(x),\phi(\sigma)) = \|g(x)-\phi(\sigma)\|^2_{\mathcal{F}}.$$

$\Longrightarrow \widehat{g} : \mathcal{X} \to \mathcal{F}$

- **Step 2 - Pre-image problem**: solve, for any x in $\mathcal{X}$, the pre-image problem that provides a prediction in the original space $\mathfrak{S}_n$:

$$\widehat{s}(x) = \underset{\sigma\in\mathfrak{S}_n}{\text{argmin}}\,\|\phi(\sigma)-\widehat{g}(x)\|^2_{\mathcal{F}}$$

Ranking Embeddings

[Ciliberto et al., 2016] have proven consistency results under some assumptions on the loss Δ/the mapping ϕ, which apply to:

- Kendall's τ distance:

$$\Delta_\tau(\sigma, \sigma') = \sum_{i<j} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\}$$

$$\rightarrow \phi(\sigma) = \begin{pmatrix} \vdots \\ sign(\sigma(i) - \sigma(j)) \\ \vdots \end{pmatrix}_{1 \leq i < j \leq n} \in \mathbb{R}^{n(n-1)/2}$$

- Hamming distance:

$$\Delta_H(\sigma, \sigma') = \sum_{i=1}^{n} \mathbb{I}[\sigma(i) \neq \sigma'(i)].$$

$$\rightarrow \phi(\sigma) = (\mathbb{I}\{\sigma(i) = j\})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$$

- **consistency holds, but still the pre-image problem is hard**

Structured prediction results

Table 2: Mean Kendall's tau coefficient on benchmark datasets

	authorship	glass	iris	vehicle	vowel	wine
kNN Kemeny	**0.94**±0.02	0.85±0.06	0.95±0.05	0.85±0.03	0.85±0.02	0.94±0.05
kNN Lehmer	0.93±0.02	0.85±0.05	0.95±0.04	0.84±0.03	0.78±0.03	0.94±0.06
ridge Hamming	-0.00±0.02	0.08±0.05	-0.10±0.13	-0.21±0.03	0.26±0.04	-0.36±0.03
ridge Lehmer	0.92±0.02	0.83±0.05	**0.97**±0.03	0.85±0.02	0.86±0.01	0.84±0.08
ridge Kemeny	**0.94**±0.02	0.86±0.06	**0.97**±0.05	**0.89**±0.03	**0.92**±0.01	0.94±0.05
Cheng PL	**0.94**±0.02	0.84±0.07	0.96±0.04	0.86±0.03	0.85±0.02	**0.95**±0.05
Cheng LWD	0.93±0.02	0.84±0.08	0.96±0.04	0.85±0.03	0.88±0.02	0.94±0.05
Zhou RF	0.91	**0.89**	**0.97**	0.86	0.87	**0.95**

Outline

Conclusion

Ranking data presents great and interesting challenges:

- Most of the maths from euclidean spaces cannot be applied
- But our intuitions still hold
- Based on our results on ranking aggregation, we develop a novel approach to ranking regression/label ranking
- Our contributions: theoretical results for this problem and new algorithms

Openings:

- How to extend to incomplete rankings (+with ties)?

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US General Social Survey

Participants were asked to rank 5 aspects about a job: ”high income”, ”no danger of being fired”, ”short working hours”, ”chances for advancement”, ”work important and gives a feeling of accomplishment”.

- 18544 samples collected between 1973 and 2014.
- 8 individual attributes are considered: sex, race, birth cohort, highest educational degree attained, family income, marital status, number of children, household size
- plus 3 attributes of work conditions: working status, employment status, and occupation.

Results:
Risk of decision tree: 2,763 $\rightarrow$ Splitting variables:

1) occupation 2) race 3) degree