Ranking Median Regression: Learning to Order through Local Consensus

Anna Korba⋆ Stephan Clémençon⋆ Eric Sibony†

⋆ Telecom ParisTech, † Shift Technology

CAp 2018

Outline

1. Ranking Regression

2. Background on Ranking Aggregation/Medians

3. Risk Minimization for Ranking (Median) Regression

4. Algorithms - Local Median Methods

Outline

Ranking Regression

Background on Ranking Aggregation/Medians

Risk Minimization for Ranking (Median) Regression

Algorithms - Local Median Methods

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \ldots, n\}$ (Ex: $\{1, 2, 3, 4\}$)

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \dots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \ldots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

Ex: $\sigma(2) = 1, \sigma(1) = 2, \cdots \Rightarrow \sigma = 2134$

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \ldots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

  Ex: $\sigma(2) = 1, \sigma(1) = 2, \cdots \Rightarrow \sigma = 2134$
- $\mathfrak{S}_n$: set of permutations of $[\![n]\!]$, the symmetric group.

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \ldots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

  Ex: $\sigma(2) = 1, \sigma(1) = 2, \cdots \Rightarrow \sigma = 2134$
- $\mathfrak{S}_n$: set of permutations of $[\![n]\!]$, the symmetric group.

Problem: Given a vector X (e.g, the characteristics of an individual), the goal is to predict (her preferences) as a random permutation Σ in $\mathfrak{S}_n$.

Ranking Regression

Consider:

- A set of n items: $[\![n]\!] = \{1, \ldots, n\}$ (Ex: $\{1, 2, 3, 4\}$)
- A individual expresses her preferences as (full) ranking, i.e a strict order $\succ$ over n :

$$a_1 \succ a_2 \succ \cdots \succ a_n \quad (\text{Ex: } 2 \succ 1 \succ 3 \succ 4)$$

- Also seen as the permutation σ that maps an item to its rank:

$$a_1 \succ \cdots \succ a_n \quad \Leftrightarrow \quad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

  Ex: $\sigma(2) = 1, \sigma(1) = 2, \cdots \Rightarrow \sigma = 2134$
- $\mathfrak{S}_n$: set of permutations of $[\![n]\!]$, the symmetric group.

Problem: Given a vector X (e.g, the characteristics of an individual), the goal is to predict (her preferences) as a random permutation Σ in $\mathfrak{S}_n$.

 $\Longrightarrow$  $\succ$  $\succ$  $\succ$ 

Related Work

- Has been referred to as **label ranking** in the literature [Tsoumakas et al., 2009], [Vembu and Gärtner, 2010]
- Related to multiclass and multilabel classification
- A lot of applications, e.g : document categorization, meta-learning
  - rank a set of topics relevant for a given document
  - rank a set of algorithms according to their suitability for a new dataset, based on the characteristics of the dataset
- A lot of approaches rely on parametric modelling [Cheng and Hüllermeier, 2009], [Cheng et al., 2009], [Cheng et al., 2010]

Related Work

- Has been referred to as **label ranking** in the literature [Tsoumakas et al., 2009], [Vembu and Gärtner, 2010]
- Related to multiclass and multilabel classification
- A lot of applications, e.g : document categorization, meta-learning
  - rank a set of topics relevant for a given document
  - rank a set of algorithms according to their suitability for a new dataset, based on the characteristics of the dataset
- A lot of approaches rely on parametric modelling [Cheng and Hüllermeier, 2009], [Cheng et al., 2009], [Cheng et al., 2010]

$\Rightarrow$ We develop an approach free of any parametric assumptions (**local learning**) relying on results and framework developped in [Korba et al., 2017] for **ranking aggregation**.

Our Problem

Suppose we observe $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$ i.i.d. copies of the pair $(X,\ \Sigma)$, where

- $X \sim \mu$, where μ is a distribution on some feature space $\mathcal{X}$
- $\Sigma \sim P_X$, where P_X is the conditional probability distribution (on $\mathfrak{S}_n$): $P_X(\sigma) = \mathbb{P}[\Sigma = \sigma | X]$

Ex: Users i with characteristics X_i order items by preference resulting in Σ_i.

Goal: Learn a predictive ranking rule :

$$\begin{array}{rccc} s\ : & \mathcal{X} & \rightarrow & \mathfrak{S}_n \\ & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Objective

Performance: Measured by the risk:

$$\mathcal{R}(s) = \mathbb{E}_{X \sim \mu, \Sigma \sim P_X} \left[d_\tau \left(s(X), \Sigma \right) \right]$$

Objective

Performance: Measured by the risk:

$$\mathcal{R}(s) = \mathbb{E}_{X \sim \mu, \Sigma \sim P_X} \left[d_\tau \left(s(X), \Sigma \right) \right]$$

where d is the Kendall's tau distance, i.e. for $\sigma, \sigma' \in \mathfrak{S}_n$:

$$d_\tau(\sigma, \sigma') = \sum_{1 \leq i < j \leq n} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\},$$

Ex: σ= 1234, σ'= 2413 $\Rightarrow d_\tau(\sigma, \sigma') = 3$ (disagree on (12),(14),(34)).

Piecewise Constant Ranking Rules

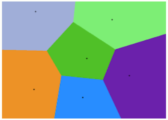
Our approach: build *piecewise constant* ranking rules, i.e:
Ranking rules that are constant on each cell of a partition of $\mathcal{X}$ built from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

Piecewise Constant Ranking Rules

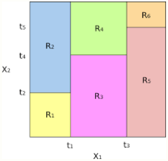
Our approach: build *piecewise constant* ranking rules, i.e:
Ranking rules that are constant on each cell of a partition of $\mathcal{X}$ built from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

Two methods are investigated:

- k-nearest neighbor (Voronoi partitioning)

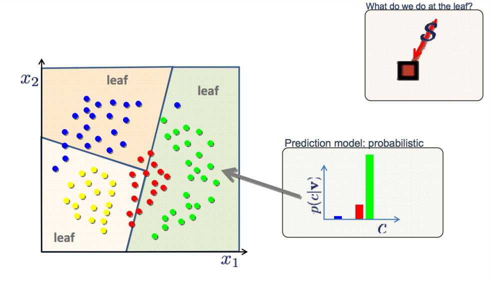


- decision tree (Recursive partitioning)



Compute Local Labels/Medians

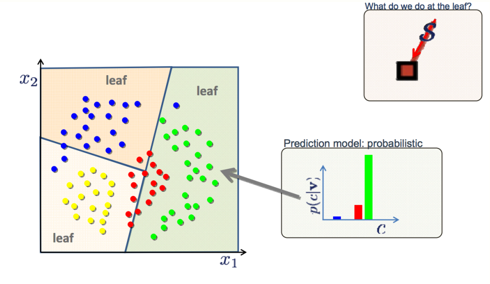
For classification, the label of a cell (ex: a leaf) is the **majority** label among the training data which fall in this cell.



Ex: 4 classes; green, red, blue, yellow $\rightarrow$ green will be the label for the right cell.

Compute Local Labels/Medians

For classification, the label of a cell (ex: a leaf) is the **majority** label among the training data which fall in this cell.



Ex: 4 classes; green, red, blue, yellow $\rightarrow$ green will be the label for the right cell.

Problem: Our labels are *permutations* σ:

For a cell $\mathcal{C}$, if $\sigma_1, \ldots, \sigma_N \in \mathcal{C}$, how do we aggregate them into a final label σ^*?

$\Longrightarrow$ Ranking aggregation problem.

Outline

Ranking Regression

Background on Ranking Aggregation/Medians

Risk Minimization for Ranking (Median) Regression

Algorithms - Local Median Methods

Ranking Aggregation - Methods

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

Ranking Aggregation - Methods

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

Kemeny's rule (1959) - Optimization pb

$$\text{Solve} \quad \sigma^* = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}} \sum_{k=1}^{N} d(\sigma, \sigma_k)$$

Ranking Aggregation - Methods

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

Kemeny's rule (1959) - Optimization pb

$$\text{Solve} \quad \sigma^* = \operatorname*{argmin}_{\sigma \in \mathfrak{S}_n} \sum_{k=1}^{N} d(\sigma, \sigma_k)$$

Problem: NP-hard.

Ranking Aggregation - Methods

Suppose we have a dataset of rankings/permutations $\mathcal{D}_N = (\sigma_1, \ldots, \sigma_N) \in \mathfrak{S}_n^N$. We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

Kemeny's rule (1959) - Optimization pb

$$\text{Solve} \quad \sigma^* = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}} \sum_{k=1}^{N} d(\sigma, \sigma_k)$$

Problem: NP-hard.

Copeland method - Scoring method

Sort the items i according to their Copeland score s_C:

$$s_C(i) = \frac{1}{N} \sum_{k=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{I}[\sigma_k(i) < \sigma_k(j)]$$

which counts the number of pairwise victories of item i over the other items $j \neq i \Rightarrow \mathcal{O}(n^2 N)$ complexity.

Statistical Ranking Aggregation [Korba et al., 2017]

Probabilistic Modeling

$$\mathcal{D}_N = (\Sigma_1, \ldots, \Sigma_N) \qquad \text{with} \qquad \Sigma_k \sim P$$

where P distribution on $\mathfrak{S}_n$.

Statistical Ranking Aggregation [Korba et al., 2017]

Probabilistic Modeling

$$\mathcal{D}_N = (\Sigma_1, \dots, \Sigma_N) \qquad \text{with} \qquad \Sigma_k \sim P$$

where P distribution on $\mathfrak{S}_n$.

Definition

A **Kemeny median** of P is solution of:

$$\sigma^*_P = \underset{\sigma \in \mathfrak{S}_n}{\text{argmin}}\, L_P(\sigma), \tag{1}$$

where $L_P(\sigma) = \mathbb{E}_{\Sigma \sim P}[d(\sigma, \Sigma)]$ is **the risk** of σ.

Question: Can we exhibit some conditions on P so that solving (1) is tractable?

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Strict Stochastic Transitivity (**SST**): $(p_{i,j} \neq 1/2 \ \forall i, j)$

$p_{i,j} > 1/2$ and $p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Strict Stochastic Transitivity (**SST**): ($p_{i,j} \neq 1/2 \; \forall i, j$)

$$p_{i,j} > 1/2 \text{ and } p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$$

Low-Noise/**NA**(h) for $h > 0$ ([Audibert and Tsybakov, 2007]):

$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Strict Stochastic Transitivity (**SST**): ($p_{i,j} \neq 1/2 \ \forall i, j$)

$$p_{i,j} > 1/2 \text{ and } p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$$

Low-Noise/**NA**(h) for $h > 0$ ([Audibert and Tsybakov, 2007]):

$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

Our result

Suppose P satisfies **SST and NA**(h) for a given $h > 0$. Then with overwhelming probability $1 - \frac{n(n-1)}{4} e^{-\alpha_h N}$:

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Strict Stochastic Transitivity (**SST**): ($p_{i,j} \neq 1/2 \ \forall i, j$)

$$p_{i,j} > 1/2 \text{ and } p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$$

Low-Noise/**NA**(h) for $h > 0$ ([Audibert and Tsybakov, 2007]):

$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

Our result

Suppose P satisfies **SST and NA**(h) for a given $h > 0$. Then with overwhelming probability $1 - \frac{n(n-1)}{4} e^{-\alpha_h N}$:

$\widehat{P}$ also verifies **SST**...

Exact Solutions [Korba et al., 2017]

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ prob. that item $i \succ j$ (is preferred to).

Strict Stochastic Transitivity (**SST**): ($p_{i,j} \neq 1/2 \; \forall i,j$)

$$p_{i,j} > 1/2 \text{ and } p_{j,k} > 1/2 \Rightarrow p_{i,k} > 1/2.$$

Low-Noise/**NA**(h) for $h > 0$ ([Audibert and Tsybakov, 2007]):

$$\min_{i<j} |p_{i,j} - 1/2| \geq h.$$

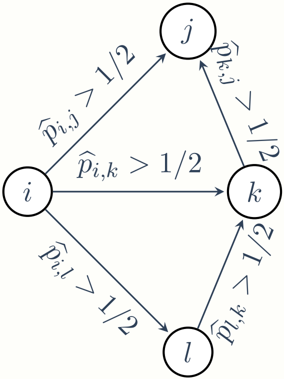
Our result

Suppose P satisfies **SST and NA**(h) for a given $h > 0$. Then with overwhelming probability $1 - \frac{n(n-1)}{4} e^{-\alpha_h N}$:

$\widehat{P}$ also verifies **SST**...and the Kemeny median of P is given by the empirical Copeland ranking:

$$\sigma^*_P(i) = 1 + \sum_{j \neq i} \mathbb{I}\{\widehat{p}_{i,j} < \frac{1}{2}\} \quad \text{for } 1 \leq i \leq n$$

Graph of pairwise probabilities



$$\sigma^*_P(i) = 1 + \sum_{j \neq i} \mathbb{I}\{\widehat{p}_{i,j} < \frac{1}{2}\} \quad \text{for } 1 \leq i \leq n$$

$\Rightarrow$ sort the i's by increasing input degree

Outline

Ranking Regression

Background on Ranking Aggregation/Medians

Risk Minimization for Ranking (Median) Regression

Algorithms - Local Median Methods

Our Problem - Ranking Regression

Goal: Learn a predictive ranking rule :

$$\begin{array}{rcccc} s & : & \mathcal{X} & \rightarrow & \mathfrak{S}_n \\ & & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Performance: Measured by the risk:

$$\begin{aligned} \mathcal{R}(s) &= \mathbb{E}_{X\sim\mu,\Sigma\sim P_X}\left[d_\tau\left(s(X),\Sigma\right)\right] \\ &= \mathbb{E}_{X\sim\mu}\left[\mathbb{E}_{\Sigma\sim P_X}\left[d_\tau\left(s(X),\Sigma\right)\right]\right] \\ &= \mathbb{E}_{X\sim\mu}\left[L_{P_X}(s(X))\right] \end{aligned}$$

Our Problem - Ranking Regression

Goal: Learn a predictive ranking rule :

$$\begin{array}{ccccc} s & : & \mathcal{X} & \to & \mathfrak{S}_n \\ & & x & \mapsto & s(x) \end{array}$$

which given a random vector X, predicts the permutation Σ on the n items.

Performance: Measured by the risk:

$$\begin{aligned} \mathcal{R}(s) &= \mathbb{E}_{X\sim\mu,\Sigma\sim P_X}\left[d_\tau\left(s(X),\Sigma\right)\right] \\ &= \mathbb{E}_{X\sim\mu}\left[\mathbb{E}_{\Sigma\sim P_X}\left[d_\tau\left(s(X),\Sigma\right)\right]\right] \\ &= \mathbb{E}_{X\sim\mu}\left[L_{P_X}(s(X))\right] \end{aligned}$$

$\Rightarrow$ Ranking regression is an extension of ranking aggregation.

Optimal Elements and Relaxation

Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \text{argmin}_{\sigma \in \mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

Optimal Elements and Relaxation

Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \text{argmin}_{\sigma \in \mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

Optimal elements

The predictors s^* minimizing $\mathcal{R}(s)$ are the ones that maps any point $X \in \mathcal{X}$ to the **conditional** Kemeny median of P_X:

$$\boxed{s^* = \underset{s \in \mathcal{S}}{\text{argmin}}\, \mathcal{R}(s) \quad \Leftrightarrow \quad s^*(X) = \sigma^*_{P_X}}$$

Optimal Elements and Relaxation

Assumption

For $X \in \mathcal{X}$, P_X is **SST**: $\Rightarrow \sigma^*_{P_X} = \text{argmin}_{\sigma \in \mathfrak{S}_n} L_{P_X}(\sigma)$ is **unique**.

Optimal elements

The predictors s^* minimizing $\mathcal{R}(s)$ are the ones that maps any point $X \in \mathcal{X}$ to the **conditional** Kemeny median of P_X:

$$s^* = \underset{s \in \mathcal{S}}{\text{argmin}}\, \mathcal{R}(s) \quad \Leftrightarrow \quad s^*(X) = \sigma^*_{P_X}$$

To minimize the risk $\mathcal{R}(s)$ approximately:

$$\sigma^*_{P_X} \text{ for any } X \Longrightarrow \sigma^*_{P_{\mathcal{C}}} \text{ for any } X \in \mathcal{C}$$

where $P_{\mathcal{C}}(\sigma) = \mathbb{P}[\Sigma = \sigma | X \in \mathcal{C}]$.

$\Rightarrow$ We develop Local consensus methods.

Statistical Framework- ERM

Optimize a statistical version of the theoretical risk based on the training data (X_k, Σ_k)'s:

$$\min_{s \in \mathcal{S}} \widehat{\mathcal{R}}_N(s) = \frac{1}{N} \sum_{k=1}^{N} d_\tau(s(X_k),\ \Sigma_k)$$

where $\mathcal{S}$ is the set of measurable mappings.

Statistical Framework- ERM

Optimize a statistical version of the theoretical risk based on the training data (X_k, Σ_k)'s:

$$\min_{s \in \mathcal{S}} \widehat{\mathcal{R}}_N(s) = \frac{1}{N} \sum_{k=1}^{N} d_\tau(s(X_k),\ \Sigma_k)$$

where $\mathcal{S}$ is the set of measurable mappings.

$\Rightarrow$ We consider a subset $\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}$:

- rich enough so that $\inf_{s \in \mathcal{S}_{\mathcal{P}}} \mathcal{R}(s) - \inf_{s \in \mathcal{S}} \mathcal{R}(s)$ is "small"
- ideally appropriate for greedy optimization.

$\Rightarrow$ $\mathcal{S}_{\mathcal{P}}$= space of piecewise constant ranking rules ("local consensus methods")

Our results

Rates of convergence

- classical rates $\mathcal{O}(1/\sqrt{N})$ for ERM.
- fast rates $\mathcal{O}(1/N)$ under a "uniform" **NA**(h).

Approximation Error

Suppose that:

There exists $M < \infty$ such that:

$\forall (x, x') \in \mathcal{X}^2, \;\; \sum_{i<j} |p_{i,j}(x) - p_{i,j}(x')| \leq M \cdot ||x - x'||$.

Then:

$$\boxed{\mathcal{R}(s^*_{\mathcal{P}}) - \mathcal{R}(s^*) \leq M.\delta_{\mathcal{P}}}$$

where $\delta_{\mathcal{P}}$ is the max. diameter of $\mathcal{P}$'s cells.

Outline

Ranking Regression

Background on Ranking Aggregation/Medians

Risk Minimization for Ranking (Median) Regression

Algorithms - Local Median Methods

Partitioning Methods

Goal: Generate partitions $\mathcal{P}_N$ from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

For $\mathcal{C} \in \mathcal{P}_N$, consider its empirical distribution:

$$\widehat{P}_{\mathcal{C}} = \frac{1}{N_{\mathcal{C}}} \sum_{k:X_k \in \mathcal{C}} \delta_{\Sigma_k}$$

and compute locally its Empirical Kemeny median $\widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}$.

Partitioning Methods

Goal: Generate partitions $\mathcal{P}_N$ from the training data $(X_1, \Sigma_1), \ldots, (X_N, \Sigma_N)$.

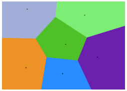
For $\mathcal{C} \in \mathcal{P}_N$, consider its empirical distribution:

$$\widehat{P}_{\mathcal{C}} = \frac{1}{N_{\mathcal{C}}} \sum_{k : X_k \in \mathcal{C}} \delta_{\Sigma_k}$$

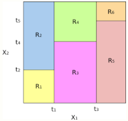
and compute locally its Empirical Kemeny median $\widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}$.

Two methods are investigated:

- k-nearest neighbor (Voronoi partitioning)



- decision tree (Recursive partitioning)



K-Nearest Neigbors Algorithm

1. Fix $k \in \{1, \ \dots, \ N\}$ and a query point $x \in \mathcal{X}$
2. Sort $(X_1, \Sigma_1), \dots, (X_N, \Sigma_N)$ by increasing order of the distance to x : $\|X_{(1,N)} - x\| \leq \dots \leq \|X_{(N,N)} - x\|$
3. Consider next the empirical distribution calculated using the k training points closest to x

$$\widehat{P}(x) = \frac{1}{k} \sum_{l=1}^{k} \delta_{\Sigma_{(l,N)}}$$

and compute the pseudo-empirical Kemeny median, yielding the k-NN prediction at x:

$$s_{k,N}(x) \overset{def}{=} \widetilde{\sigma}^*_{\widehat{P}(x)}.$$

$\Rightarrow$ We recover the classical bound $\mathcal{R}(s_{k,N}) - \mathcal{R}^* = \mathcal{O}(\frac{1}{\sqrt{k}} + \frac{k}{N})$

Decision Tree

Split recursively the feature space $\mathcal{X}$ to minimize some impurity criterion.

Analog to Gini criterion in multiclassification: m classes, f_i proportion of class $i \to I_G(f) = \sum_{i=1}^{m} f_i(1 - f_i)$

Decision Tree

Split recursively the feature space $\mathcal{X}$ to minimize some impurity criterion.

Analog to Gini criterion in multiclassification: m classes, f_i proportion of class $i \rightarrow I_G(f) = \sum_{i=1}^{m} f_i(1-f_i)$

Here, for a cell $\mathcal{C} \in \mathcal{P}_N$:

- Impurity [Alvo and Philip, 2014]:

$$\gamma_{\widehat{P}_{\mathcal{C}}} = \frac{1}{2} \sum_{i<j} \widehat{p}_{i,j}(\mathcal{C}) \left(1 - \widehat{p}_{i,j}(\mathcal{C})\right)$$

  which is tractable and satisfies the double inequality

$$\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}} \leq \min_{\sigma \in \mathfrak{S}_n} L_{\widehat{P}_{\mathcal{C}}}(\sigma) \leq 2\widehat{\gamma}_{\widehat{P}_{\mathcal{C}}}.$$

- Terminal value : Compute the pseudo-empirical median of a cell $\mathcal{C}$:

$$s_{\mathcal{C}}(x) \overset{def}{=} \widetilde{\sigma}^*_{\widehat{P}_{\mathcal{C}}}.$$

Conclusion

Interesting challenges:

- Most of the maths from euclidean spaces cannot be applied, but our insights still hold
- Based on our results on ranking aggregation, we develop a novel approach to ranking regression/label ranking
- Theoretical guarantees
- We propose two practical algorithms

Openings: How to extend to incomplete rankings (+with ties)?

Alvo, M. and Philip, L. (2014).
Decision tree models for ranking data.
In *Statistical Methods for Ranking Data*, pages 199–222.
Springer.

Audibert, J.-Y. and Tsybakov, A. (2007).
Fast learning rates for plug-in classifiers.
Annals of statistics, 35(2):608–633.

Cheng, W., Dembczyński, K., and Hüllermeier, E. (2010).
Label ranking methods based on the Plackett-Luce model.
In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, pages 215–222.

Cheng, W., Hühn, J., and Hüllermeier, E. (2009).
Decision tree and instance-based learning for label ranking.
In *Proceedings of the 26th International Conference on Machine Learning (ICML-09)*, pages 161–168.

Cheng, W. and Hüllermeier, E. (2009).

A new instance-based label ranking approach using the mallows model.
Advances in Neural Networks–ISNN 2009, pages 707–716.

Jiang, X., Lim, L.-H., Yao, Y., and Ye, Y. (2011).
Statistical ranking and combinatorial hodge theory.
Mathematical Programming, 127(1):203–244.

Korba, A., Clémençon, S., and Sibony, E. (2017).
A learning theory of ranking aggregation.
In *Proceeding of AISTATS 2017*.

Tsoumakas, G., Katakis, I., and Vlahavas, I. (2009).
Mining multi-label data.
In *Data mining and knowledge discovery handbook*, pages 667–685. Springer.

Vembu, S. and Gärtner, T. (2010).
Label ranking algorithms: A survey.
In *Preference learning*, pages 45–64. Springer.

In practice: Pseudo-empirical Kemeny Medians

- If $\widehat{P}$ is SST, compute $\sigma^*_{\widehat{P}}$ with Copeland method based on $\widehat{p}_{i,j}$

In practice: Pseudo-empirical Kemeny Medians

- If $\widehat{P}$ is SST, compute $\sigma^*_{\widehat{P}}$ with Copeland method based on $\widehat{p}_{i,j}$
- Else, compute $\widetilde{\sigma}^*_{\widehat{P}}$ with empirical Borda count ([Jiang et al., 2011])

$$\widetilde{\sigma}^*_{\widehat{P}}(i) = \frac{1}{N}\sum_{k=1}^{N} \Sigma_k(i) \quad \text{for } 1 \leq i \leq n$$

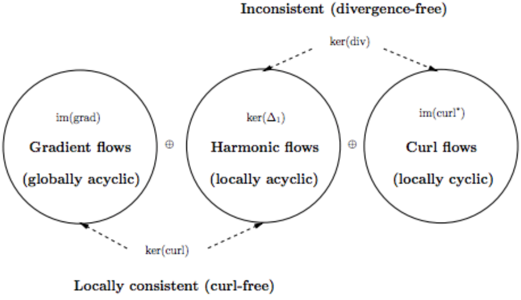


FIGURE 2. Hodge/Helmholtz decomposition of pairwise rankings

Experimental Results