A Learning Theory of Ranking Aggregation

France/Japan Machine Learning Workshop

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2. Statistical Framework
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The Ranking Aggregation Problem

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Framework

- n items: $\{1, \ldots, n\}$.
- N rankings on the n items (from most preferred to last): $i_1 \succ i_2 \succ \cdots \succ i_n$.
- $i_1 \succ \cdots \succ i_n \Longleftrightarrow$ permutation σ on $\{1, \ldots, n\}$ s.t. $\sigma(i_j) = j$.

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Consensus Ranking

We want to find a global order ("consensus") σ^* on the n items that best represents the dataset.

Copeland Rule.

Sort the items according to their Copeland score, defined for each item i by:

$$s_C(i) = \frac{1}{N} \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{I}[\sigma_t(i) < \sigma_t(j)]$$

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Kemeny's rule (1959).

Find the solution of :

$$min_{\sigma \in \mathfrak{S}_n} \sum_{i=1}^{N} d(\sigma, \sigma_t) \tag{1}$$

where d is the Kendall's tau distance:

$$d_\tau(\sigma, \sigma') = \sum_{i<j} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\},$$

Kemeny's consensus has a lot of interesting properties, but it is NP-hard to compute, even for $n = 4$ (see Dwork et al., 2001).

Statistical Framework

Statistical Reformulation

Suppose the dataset is composed of N i.i.d. copies $\Sigma_1, \dots, \Sigma_N$ of a r.v. $\Sigma \sim P$. A (Kemeny) **median** of P w.r.t. d is solution of:

$$\boxed{\min_{\sigma \in \mathfrak{S}_n} \mathbb{E}_{\Sigma \sim P}[d(\Sigma, \sigma)],}$$

where $L(\sigma) = \mathbb{E}_{\Sigma \sim P}[d(\Sigma, \sigma)]$ is **the risk** of σ.

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Let $\widehat{L}_N(\sigma) = \frac{1}{N} \sum_{t=1}^{N} d(\Sigma_t, \sigma)$.

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Goal of our analysis:

Study the performance of **Kemeny empirical medians**, i.e. solutions $\widehat{\sigma}_N$ of:

$$\min_{\sigma \in \mathfrak{S}_n} \widehat{L}_N(\sigma),$$

through the excess of risk $L(\widehat{\sigma}_N) - L^*$.

$\Rightarrow$ We establish links with Copeland method.

Risk of Ranking Aggregation

The risk of a median σ is $L(\sigma) = \mathbb{E}_{\Sigma \sim P}[d(\Sigma, \sigma)]$, where d is the Kendall's tau distance:

$$d(\sigma, \sigma') = \sum_{\{i,j\} \subset [\![n]\!]} \{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\}$$

Let $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ the probability that item i is preferred to item j.

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The risk can be rewritten:

$$L(\sigma) \; = \; \sum_{i<j} p_{i,j} \mathbb{I}\{\sigma(i) \; > \; \sigma(j)\} + \sum_{i<j} (1 \, - \, p_{i,j}) \mathbb{I}\{\sigma(i) \; < \; \sigma(j)\}.$$

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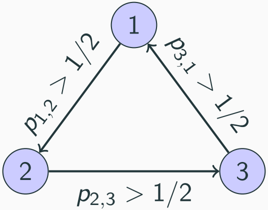
So if there exists a permutation σ verifying: $\forall i < j$ s.t. $p_{i,j} \neq 1/2$,

$$(\sigma(j) - \sigma(i)) \cdot (p_{i,j} - 1/2) > 0, \tag{2}$$

it would be necessary of median for P.

Preference cycles



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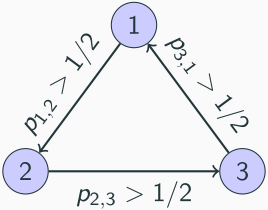

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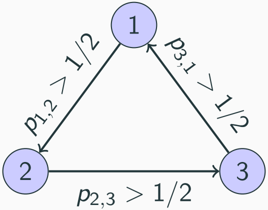

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Definition

P on $\mathfrak{S}_n$ is **stochastically transitive** if : $\forall (i, j, k) \in [\![n]\!]^3$,

$$p_{i,j} \geq 1/2 \text{ and } p_{j,k} \geq 1/2 \Rightarrow p_{i,k} \geq 1/2.$$

Moreover, if $p_{i,j} \neq 1/2$ for all $i < j$, P is **strictly stochastically transitive**.

Results

Optimality

Theorem

- If P is stochastically transitive, there exists $\sigma^* \in \mathfrak{S}_n$ verifying: $\forall i < j$ s.t. $p_{i,j} \neq 1/2$,

$$(\sigma(j) - \sigma(i)) \cdot (p_{i,j} - 1/2) > 0,$$

is verified.

- The **Copeland score** of an item i, that is:

$$\boxed{s^*(i) = 1 + \sum_{k \neq i} \mathbb{I}\{p_{i,k} < \frac{1}{2}\}}$$

defines a permutation $s^* \in \mathfrak{S}_n$ and is the unique median of P iff P is strictly stochastically transitive.

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Universal Rates

The excess risk of $\widehat{\sigma}_N$ is upper bounded:

(i) In expectation by

$$\mathbb{E}\left[L(\widehat{\sigma}_N) - L^*\right] \leq \frac{n(n-1)}{2\sqrt{N}}$$

(ii) With probability higher than $1-\delta$ for any $\delta \in (0,1)$ by

$$L(\widehat{\sigma}_N) - L^* \leq \frac{n(n-1)}{2}\sqrt{\frac{2\log(n(n-1)/\delta)}{N}}.$$

Conditions for Fast Rates

Suppose that P verifies:

- the Stochastic Transitivity condition:

$$p_{i,j} \geq 1/2 \text{ and } p_{j,k} \geq 1/2 \Rightarrow p_{i,k} \geq 1/2.$$

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$\Rightarrow$ Introduced for binary classification (see Koltchinskii and Beznosova, 2005).
$\Rightarrow$ Used for estimation of matrix of pairwise probabilities (see Shah et al., 2016).

Fast rates

Let $\alpha_h = \frac{1}{2}\log\left(1/(1-4h^2)\right)$.

Assume that P satisfies the previous conditions.

(i) For any empirical Kemeny median $\widehat{\sigma}_N$, we have:

$$\mathbb{E}\left[L(\widehat{\sigma}_N) - L^*\right] \leq \frac{n^2(n-1)^2}{8} e^{-\alpha_h N}.$$

(ii) With probability at least $1 - (n(n-1)/4)e^{-\alpha_h N}$, the empirical Copeland score

$$\widehat{s}_N(i) = 1 + \sum_{k \neq i} \mathbb{I}\{\widehat{p}_{i,k} < \frac{1}{2}\}$$

for $1 \leq i \leq n$ belongs to $\mathfrak{S}_n$ and is the unique solution of Kemeny empirical minimization.

$\Rightarrow$ In practice: under the needed conditions, Copeland method ($\mathcal{O}(N\binom{n}{2})$) outputs the Kemeny consensus (NP-hard) with high prob.

Conclusion and future directions

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