Sampling with Kernelized Wasserstein Gradient Flows

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IMSI Applied Optimal Transport Workshop

Outline

Sampling

Problem: Sample (=generate new examples) from a target distribution π over $\mathbb{R}^d$, whose density w.r.t. Lebesgue measure is known up to an intractable normalisation constant Z :

$$\pi(\theta) = \frac{\tilde{\pi}(\theta)}{Z}, \quad \tilde{\pi} \text{ known, } Z \text{ unknown.}$$

Main application: Bayesian inference, where π is the posterior distribution over parameters of a model.

Bayesian inference

Let $\mathcal{D} = (w_i, y_i)_{i=1}^m$ a dataset of labelled examples $(w_i, y_i) \overset{i.i.d.}{\sim} P_{\text{data}}$.
Assume an underlying model parametrized by θ, e.g. :

$$y = g(w, \theta) + \epsilon, \quad \epsilon \sim \mathcal{N}(0, I)$$

Goal: learn the best distribution over θ to fit the data.

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1. Compute the Likelihood:

$$p(\mathcal{D}|\theta) = \prod_{i=1}^{m} p(y_i|\theta, w_i) \propto \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \|y_i - g(w_i, \theta)\|^2 \right).$$

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2. Choose a prior distribution on the parameter:

$$\theta \sim p, \quad \text{e.g. } p(\theta) \propto \exp\left(-\frac{\|\theta\|^2}{2}\right).$$

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3. Bayes' rule yields:

$$\pi(\theta) := p(\theta|\mathcal{D}) = \frac{p(\mathcal{D}|\theta)p(\theta)}{Z} \qquad Z = \int_{\mathbb{R}^d} p(\mathcal{D}|\theta)p(\theta)d\theta$$

$$\text{i.e. } \pi(\theta) \propto \exp\left(-V(\theta)\right), \quad V(\theta) = \frac{1}{2}\sum_{i=1}^m \|y_i - g(w_i, \theta)\|^2 + \frac{\|\theta\|^2}{2}.$$

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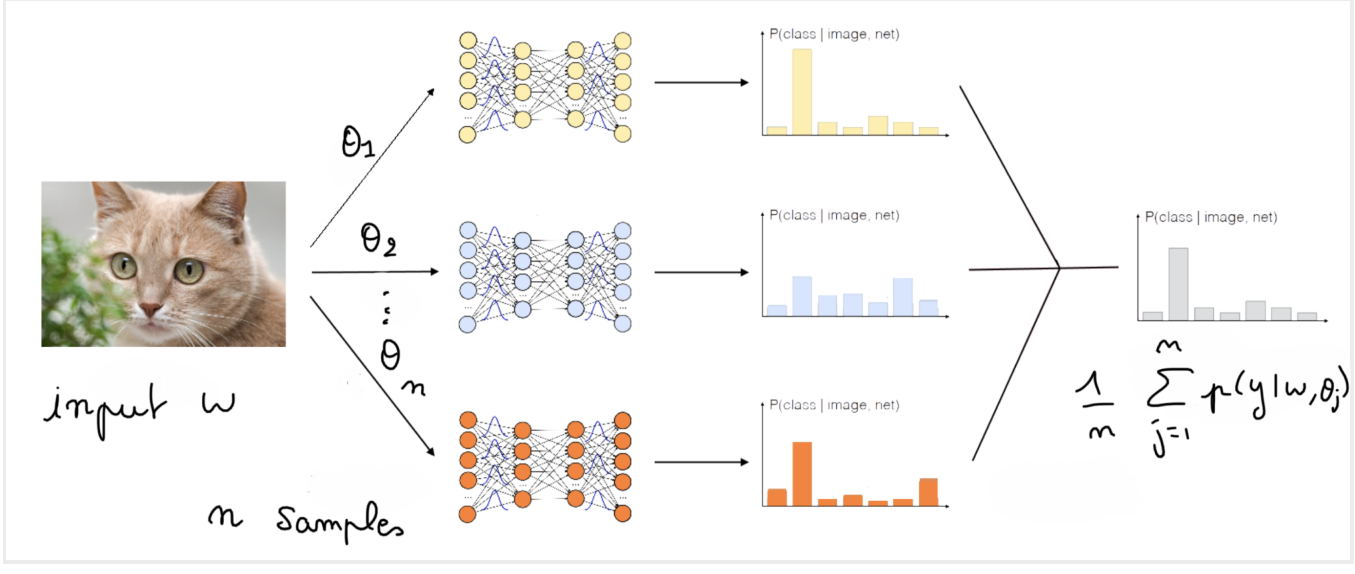

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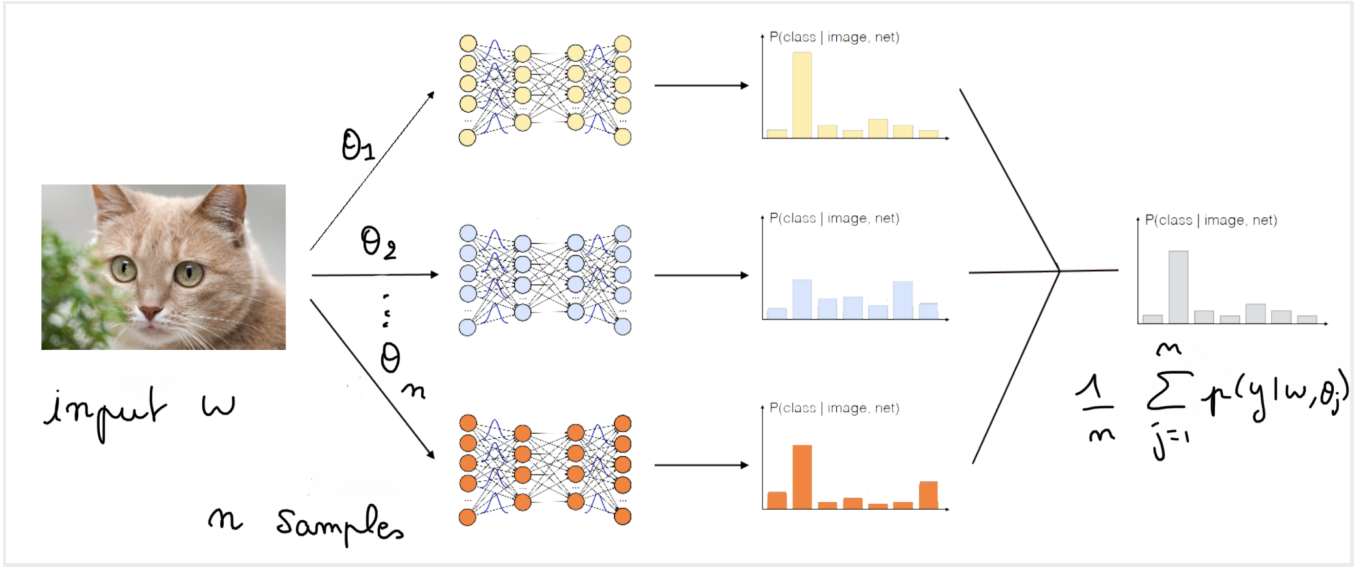

Question: how can we build μ_n?

Sampling as optimisation

Notice that

$$\pi = \operatorname*{argmin}_{\mu \in \mathcal{P}(\mathbb{R}^d)} \mathrm{KL}(\mu|\pi), \quad \mathrm{KL}(\mu|\pi) = \begin{cases} \int_{\mathbb{R}^d} \log\left(\frac{\mu}{\pi}(\theta)\right) d\mu(\theta) & \text{if } \mu \ll \pi \\ +\infty & \text{else.} \end{cases}$$

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Two (non parametric) ways to produce an approximation μ_n:

1. Markov Chain Monte Carlo (MCMC) methods: generate a Markov chain whose law converges to $\pi \propto \exp(-V)$

   Example: Langevin Monte Carlo (LMC), discretizes an overdamped Langevin diffusion

   $$d\theta_t = -\nabla V(\theta_t) + \sqrt{2} dB_t \Longrightarrow \theta_{l+1} = \theta_l - \gamma \nabla V(\theta_l) + \sqrt{2\gamma}\epsilon_l, \ \epsilon_l \sim \mathcal{N}(0, I_d)$$

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2. Interacting particle systems, e.g. by considering other metrics or functionals

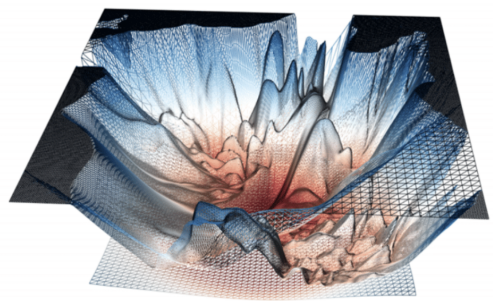
Difficult cases : non-convex potentials

Recall that

$$\pi(\theta) \propto \exp\left(-V(\theta)\right), \quad V(\theta) = \underbrace{\sum_{i=1}^{m} \|y_i - g(w_i, \theta)\|^2}_{\text{loss}} + \frac{\|\theta\|^2}{2}.$$

- if V is convex (e.g. $g(w, \theta) = \langle w, \theta \rangle$) many sampling methods are known to work quite well, including LMC
- but if its not (e.g. $g(w, \theta)$ is a neural network), the situation is much more delicate
- MCMC methods do not scale and require too many iterations

  [Izmailov et al., 2021]



A highly nonconvex loss surface, as is common in deep neural nets. From `https://www.telesens.co/2019/01/16/neural-network-loss-visualization`.

Sampling as optimization over distributions

Assume that $\pi \in \mathcal{P}_2(\mathbb{R}^d) = \left\{\mu \in \mathcal{P}(\mathbb{R}^d), \int \|x\|^2 d\mu(x) < \infty\right\}$.
We equip $\mathcal{P}_2(\mathbb{R}^d)$ with the Wasserstein-2 distance:

$$W_2^2(\nu, \mu) = \inf_{s \in \Gamma(\nu,\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 \, ds(x, y) \qquad \forall \nu, \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

where $\Gamma(\nu, \mu)$ is the set of possible couplings between ν and μ.

The sampling task can be recast as an optimization problem:

$$\pi = \operatorname*{argmin}_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \mathcal{F}(\mu), \quad \mathcal{F}(\mu) := D(\mu|\pi)$$

where D is a **dissimilarity functional** (f-div, IPM, OT distance...).

Starting from an initial distribution $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, one can then consider the **Wasserstein gradient flow** of $\mathcal{F}$ over $\mathcal{P}_2(\mathbb{R}^d)$ to transport μ_0 to π.

Outline

Euclidean gradient flow and continuity equation

Let $V : \mathbb{R}^d \to \mathbb{R}$. Consider the gradient flow

$$\frac{dX_t}{dt} = -\nabla V(x_t)$$

and assume x_0 random with density μ_0. What is the dynamics of the density μ_t of x_t ? Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a test function.

$$\frac{d}{dt}\mathbb{E}(\phi(x_t)) = -\int \langle \nabla\phi, \nabla V\rangle \mu_t(x)dx = \int \phi(x)\boldsymbol{\nabla}\cdot(\mu_t \nabla V)(x)dx,$$

and

$$\frac{d}{dt}\mathbb{E}(\phi(x_t)) = \int \phi(x)\frac{\partial \mu_t}{\partial t}(x)dx.$$

Therefore,

$$\frac{\partial \mu_t}{\partial t} = \boldsymbol{\nabla}\cdot(\mu_t \nabla V).$$

Wasserstein gradient flows (WGF) [Ambrosio et al., 2008]

The first variation of $\mu \mapsto \mathcal{F}(\mu)$ evaluated at $\mu \in \mathcal{P}$, if it exists, is the unique function $\frac{\partial \mathcal{F}(\mu)}{\partial \mu} : \mathbb{R}^d \to \mathbb{R}$ s. t. for any $\mu, \mu' \in \mathcal{P}$:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\mathcal{F}(\mu + \epsilon(\mu' - \mu)) - \mathcal{F}(\mu) \right] = \int_{\mathbb{R}^d} \frac{\partial \mathcal{F}(\mu)}{\partial \mu}(x)(d\mu' - d\mu)(x).$$

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The family $\mu : [0, \infty] \to \mathcal{P}, t \mapsto \mu_t$ satisfies a Wasserstein gradient flow of $\mathcal{F}$ if distributionally:

$$\frac{\partial \mu_t}{\partial t} = \boldsymbol{\nabla} \cdot (\mu_t \nabla_{W_2} \mathcal{F}(\mu_t)),$$

where $\nabla_{W_2} \mathcal{F}(\mu) := \nabla \frac{\partial \mathcal{F}(\mu)}{\partial \mu} \in L^2(\mu)$ denotes the Wasserstein gradient of $\mathcal{F}$.

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It can be implemented by the deterministic process:

$$\frac{dX_t}{dt} = -\nabla_{W_2} \mathcal{F}(\mu_t)(X_t)$$

Time and Space discretization - Particle system

Let $\gamma > 0$ be a step-size:

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Idea: replace it by the empirical measure of a system of n interacting particles:

$$X_0^1, \dots, X_0^n \sim \mu_0$$

and for $j = 1, \dots, n$:

$$X_{l+1}^j = X_l^j - \gamma \nabla_{W_2} \mathcal{F}(\hat{\mu}_l)(X_l^j)$$

where $\hat{\mu}_l = \frac{1}{n} \sum_{i=1}^n \delta_{X_l^j}$.

We recall that

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and that we can consider the Forward time discretisation:

$$x_{l+1} = x_l - \gamma \nabla_{W_2} \mathrm{KL}(\mu_l|\pi)(x_l), \quad x_l \sim \mu_l,$$

where $\nabla_{W_2} \mathrm{KL}(\mu_l|\pi) = \nabla \frac{\partial \mathrm{KL}(\mu_l|\pi)}{\partial \mu} = \nabla \log\big(\frac{\mu_l}{\pi}(.)\big)$.

Problem: μ_l, hence $\nabla \log(\mu_l)$ is unknown and has to be estimated from a set of particles.

Outline

Background on kernels and RKHS [Steinwart and Christmann, 2008]

- Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a positive, semi-definite kernel
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$$\mathcal{H}_k = \overline{\left\{ \sum_{i=1}^{m} \alpha_i k(\cdot, x_i);\ m \in \mathbb{N};\ \alpha_1, \ldots, \alpha_m \in \mathbb{R};\ x_1, \ldots, x_m \in \mathbb{R}^d \right\}}$$

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- It satisfies the reproducing property:

$$\forall \quad f \in \mathcal{H}_k,\ x \in \mathbb{R}^d, \quad f(x) = \langle f, k(x, .) \rangle_{\mathcal{H}_k}.$$

Stein Variational Gradient Descent [Liu and Wang, 2016]

Consider the following metric depending on k [1]

$$W_k^2(\mu_0, \mu_1) = \inf_{(\mu_t, v_t)} \left\{ \int_0^1 \|v_t\|_{\mathcal{H}_k^d}^2 dt : \frac{\partial \mu_t}{\partial t} = \nabla \cdot (\mu_t v_t) \right\}.$$

Then, the W_k gradient flow of the KL writes as the PDE [Liu, 2017], [Duncan et al., 2019]:

$$\frac{\partial \mu_t}{\partial t} + \nabla \cdot \left(\mu_t P_{\mu_t} \nabla \log \left(\frac{\mu_t}{\pi} \right) \right) = 0, \quad P_\mu : f \mapsto \int k(x, .) f(x) d\mu(x).$$

It converges to $\pi \propto \exp(-V)$ under mild conditions on k and if V grows at most polynomially [Lu et al., 2019].

[1] $W_2^2(\mu_0, \mu_1) = \inf_{(\mu_t, v_t)_{t \in [0,1]}} \left\{ \int_0^1 \|v_t\|_{L^2(\mu_t)}^2 dt : \frac{\partial \mu_t}{\partial t} = \nabla \cdot (\mu_t v_t) \right\}$.

SVGD algorithm

SVGD trick: applying the kernel integral operator to the W_2 gradient of $\mathrm{KL}(\cdot|\pi)$ leads to

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\begin{aligned}
P_\mu \nabla \log\left(\frac{\mu}{\pi}\right)(\cdot) &= \int \nabla \log\left(\frac{\mu}{\pi}\right)(x) k(x,.) d\mu(x) \\
&= \int -\nabla \log(\pi(x)) k(x,.) d\mu(x) + \int \nabla(\mu(x)) k(x,.) dx \\
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Algorithm : Starting from n i.i.d. samples $(X_0^i)_{i=1,\dots,n} \sim \mu_0$, SVGD algorithm updates the n particles as follows :

$$
\begin{aligned}
X_{l+1}^i &= X_l^i - \gamma \left[\frac{1}{n}\sum_{j=1}^n \nabla_{X_l^j} \log \pi(X_l^j) k(X_l^i, X_l^j) + \nabla_{X_l^j} k(X_l^j, X_l^i)\right] \\
&= X_l^i - \gamma P_{\mu_l^n} \nabla \log\left(\frac{\mu_l^n}{\pi}\right)(X_l^i), \quad \text{with } \mu_l^n = \frac{1}{n}\sum_{j=1}^n \delta_{X_l^j}
\end{aligned}
$$

SVGD in practice

- more than 600 citations for [Liu and Wang, 2016]
- Relative empirical success in Bayesian inference and more recently for deep networks
- It can suffer for multimodal distributions [Wenliang and Kanagawa, 2020], underestimate the target variance [Ba et al., 2021], but still can be very efficient on difficult sampling problems.

		AUROC(H)	AUROC(MD)	Accuracy	H_o/H_t	MD_o/MD_t	ECE	NLL
FashionMNIST	**Deep ensemble [38]**	0.958±0.001	0.975±0.001	91.122±0.013	6.257±0.005	6.394±0.001	**0.012±0.001**	0.129±0.001
	SVGD [46]	0.960±0.001	0.973±0.001	91.134±0.024	6.315±0.019	6.395±0.018	0.014±0.001	0.127±0.001
	f-SVGD [67]	0.956±0.001	0.975±0.001	89.884±0.015	5.652±0.009	6.531±0.005	0.013±0.001	0.150±0.001
	kde-WGD (ours)	0.960±0.001	0.970±0.001	91.238±0.019	6.587±0.019	6.379±0.018	0.014±0.001	0.128±0.001
	sge-WGD (ours)	0.960±0.001	0.970±0.001	**91.312±0.016**	6.562±0.007	6.363±0.009	**0.012±0.001**	0.128±0.001
	ssge-WGD (ours)	0.968±0.001	0.979±0.001	91.198±0.024	6.522±0.009	6.610±0.012	**0.012±0.001**	0.130±0.001
	kde-fWGD (ours)	**0.971±0.001**	**0.980±0.001**	91.260±0.011	7.079±0.016	6.887±0.015	0.015±0.001	**0.125±0.001**
	sge-fWGD (ours)	0.969±0.001	0.978±0.001	91.192±0.013	7.076±0.004	6.900±0.005	0.015±0.001	**0.125±0.001**
	ssge-fWGD (ours)	**0.971±0.001**	**0.980±0.001**	91.240±0.022	**7.129±0.006**	**6.951±0.005**	0.016±0.001	**0.124±0.001**
CIFAR10	**Deep ensemble [38]**	**0.843±0.004**	0.736±0.005	85.552±0.076	**2.244±0.006**	1.667±0.008	0.049±0.001	0.277±0.001
	SVGD [46]	0.825±0.001	0.710±0.002	85.142±0.017	2.106±0.003	1.567±0.004	0.052±0.001	0.287±0.001
	fSVGD [67]	0.783±0.001	0.712±0.001	84.510±0.031	1.968±0.004	1.624±0.003	0.049±0.001	0.292±0.001
	kde-WGD (ours)	0.838±0.001	0.735±0.004	**85.904±0.030**	2.205±0.003	1.661±0.008	0.053±0.001	**0.276±0.001**
	sge-WGD (ours)	0.837±0.003	0.725±0.004	85.792±0.035	2.214±0.010	1.634±0.004	0.051±0.001	**0.275±0.001**
	ssge-WGD (ours)	0.832±0.003	0.731±0.005	85.638±0.038	2.182±0.015	1.655±0.001	0.049±0.001	**0.276±0.001**
	kde-fWGD (ours)	0.791±0.002	**0.758±0.002**	84.888±0.030	1.970±0.004	**1.749±0.005**	**0.044±0.001**	0.282±0.001
	sge-fWGD (ours)	0.795±0.001	0.754±0.002	84.766±0.060	1.984±0.003	1.729±0.002	0.047±0.001	0.288±0.001
	ssge-fWGD (ours)	0.792±0.002	0.752±0.002	84.762±0.034	1.970±0.006	1.723±0.005	0.046±0.001	0.286±0.001

From *Repulsive Deep Ensembles are Bayesian. F. D'angelo, V. Fortuin. Conference on Neural Information Processing Systems (NeurIPS 2021).*

Continuous-time dynamics of SVGD

$$\frac{\partial \mu_t}{\partial t} + \boldsymbol{\nabla} \cdot \left(\mu_t P_{\mu_t} \nabla \log \left(\frac{\mu_t}{\pi} \right) \right) = 0, \quad P_\mu : f \mapsto \int k(x, .) f(x) d\mu(x).$$

[2] $P_\mu = S_\mu^* \circ S_\mu$, where $S_\mu : L^2(\mu) \to \mathcal{H}_k, f \mapsto \int k(x, .) f(x) d\mu(x)$ and $S_\mu^* = \iota_{\mathcal{H}_k \to L^2(\mu)}$ the injection from $\mathcal{H}_k$ to $L^2(\mu)$. We sometimes abuse notation here between P_μ, S_μ for ease of presentation.

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How fast the KL decreases along SVGD dynamics? Apply the chain rule in the Wasserstein space[2]:

$$\frac{d \, \mathrm{KL}(\mu_t | \pi)}{dt} = \left\langle V_t, \nabla \log \left(\frac{\mu_t}{\pi} \right) \right\rangle_{L^2(\mu_t)} = - \underbrace{\left\| P_{\mu_t} \nabla \log \left(\frac{\mu_t}{\pi} \right) \right\|^2_{\mathcal{H}_k}}_{\mathrm{KSD}^2(\mu_t | \pi)} \leq 0.$$

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On the r.h.s. we have the **Kernel Stein discrepancy (KSD)** [Chwialkowski et al., 2016] or **Stein Fisher information** of μ_t relative to π:

$$\begin{aligned} \left\| P_{\mu,k} \nabla \log \left(\frac{\mu}{\pi} \right) \right\|^2_{\mathcal{H}_k} &= \langle P_{\mu,k} \nabla \log \left(\frac{\mu}{\pi} \right), P_{\mu,k} \nabla \log \left(\frac{\mu}{\pi} \right) \rangle_{\mathcal{H}_k} \\ &= \iint \nabla \log \left(\frac{\mu}{\pi}(x) \right) \nabla \log \left(\frac{\mu}{\pi}(y) \right) k(x, y) d\mu(x) d\mu(y). \end{aligned}$$

Recall that the Fisher divergence is defined as $\| \nabla \log \left(\frac{\mu}{\pi} \right) \|^2_{L^2(\mu)}$.

[2] $P_\mu = S^*_\mu \circ S_\mu$, where $S_\mu : L^2(\mu) \to \mathcal{H}_k, f \mapsto \int k(x, .) f(x) d\mu(x)$ and $S^*_\mu = \iota_{\mathcal{H}_k \to L^2(\mu)}$ the injection from $\mathcal{H}_k$ to $L^2(\mu)$. We sometimes abuse notation here between P_μ, S_μ for ease of presentation.

A descent lemma in discrete time for SVGD [Korba et al., 2020]

Idea: in optimisation, descent lemmas can be shown if the objective function has a bounded Hessian.

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Assume that $\pi \propto \exp(-V)$ where $\|H_V(x)\| \leq M$.
The Hessian of the KL at μ is an operator on $L^2(\mu)$:

$$\langle f, Hess_{\mathrm{KL}(.|\pi)}(\mu) f \rangle_{L^2(\mu)} = \mathbb{E}_{X\sim\mu}\left[\langle f(X), H_V(X) f(X)\rangle + \|Jf(X)\|^2_{HS}\right]$$

and yet, this operator **is not bounded** due to the Jacobian term.

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and yet, this operator **is not bounded** due to the Jacobian term.

However: In the case of SVGD, the descent directions f are restricted to $\mathcal{H}_k$ (bounded functions, bounded derivatives for bounded $k, \nabla k$).

Proposition: Assume (boundedness of k and ∇k, H_V and moments on the trajectory), then for γ small enough:

$$\mathrm{KL}(\mu_{l+1}|\pi) - \mathrm{KL}(\mu_l|\pi) \leq -c_\gamma \underbrace{\left\| P_{\mu_l} \nabla \log\left(\frac{\mu_l}{\pi}\right)\right\|_{\mathcal{H}_k}^2}_{\mathrm{KSD}^2(\mu_l|\pi)}.$$

Rates in KSD

Consequence of the descent lemma: for γ small enough,

$$\min_{l=1,\dots,L} \mathrm{KSD}^2(\mu_l|\pi) \leq \frac{1}{L}\sum_{l=1}^{L} \mathrm{KSD}^2(\mu_l|\pi) \leq \frac{\mathrm{KL}(\mu_0|\pi)}{c_\gamma L}.$$

This result only relies on the smoothness of V, not on any kind of convexity, in contrast with many convergence results on LMC.

The KSD metrizes convergence for instance when [Gorham and Mackey, 2017]:

- π is distantly dissipative (log concave at infinity, e.g. mixture of Gaussians)
- k is the IMQ kernel defined by $k(x, y) = (c^2 + \|x - y\|_2^2)^\beta$ for $c > 0$ and $\beta \in (-1, 0)$.

Open question 1: Rates in terms of the KL objective?

To obtain rates, one may combine a descent lemma (1) of the form

$$\mathrm{KL}(\mu_{l+1}|\pi) - \mathrm{KL}(\mu_l|\pi) \leq -c_\gamma \left\| P_{\mu_n} \nabla \log\left(\frac{\mu_l}{\pi}\right) \right\|^2_{\mathcal{H}_k}$$

and the Stein log-Sobolev inequality (2) with constant λ:

$$\mathrm{KL}(\mu|\pi) \leq \frac{1}{2\lambda} \mathrm{KSD}^2(\mu|\pi) \text{ for all } \mu.$$

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Then:

$$\mathrm{KL}(\mu_{l+1}|\pi) - \mathrm{KL}(\mu_l|\pi) \underbrace{\leq}_{(1)} -c_\gamma \left\| P_{\mu_l} \nabla \log\left(\frac{\mu_n}{\pi}\right) \right\|_{\mathcal{H}_k}^2 \underbrace{\leq}_{(2)} -c_\gamma 2\lambda \, \mathrm{KL}(\mu_n|\pi).$$

Iterating this inequality yields $\mathrm{KL}(\mu_l|\pi) \leq (1 - 2c_\gamma\lambda)^l \, \mathrm{KL}(\mu_0|\pi)$.

Problem: not possible to combine (1) and (2). (2) fails to hold if k is too regular with respect to π (e.g. k bounded, π Gaussian) [Duncan et al., 2019] . Some working examples in dimension 1, open question in greater dimensions...

First Experiments (d=1)

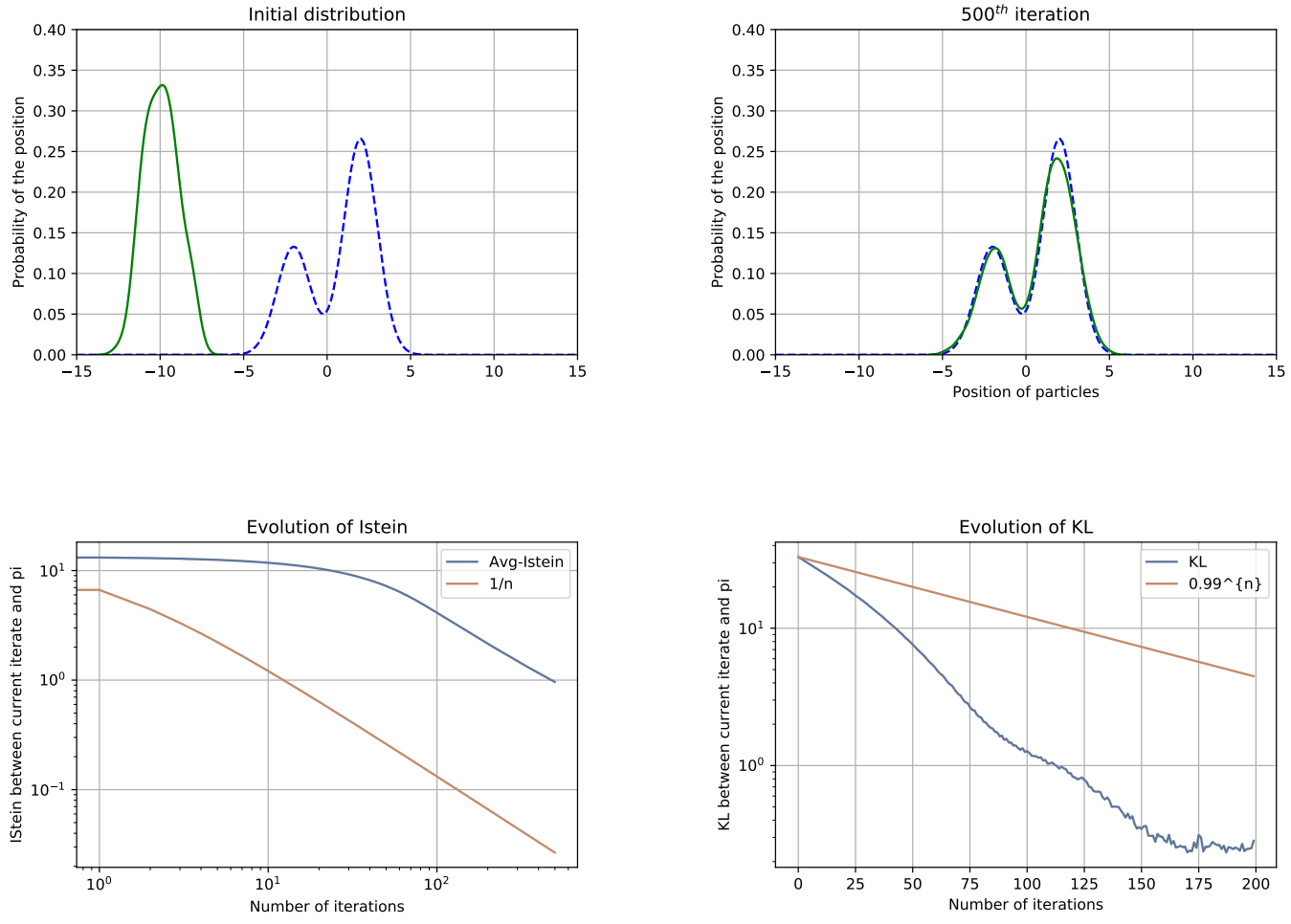
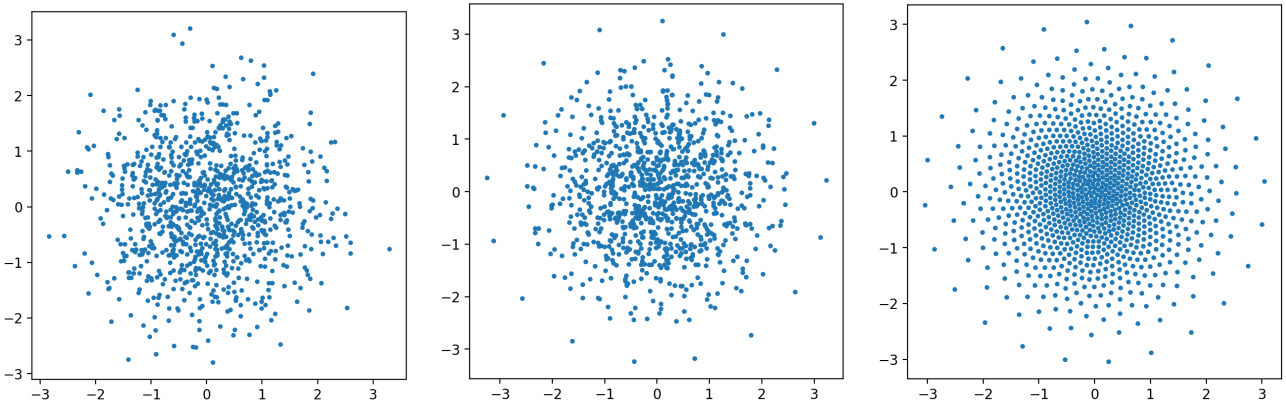


Figure: The particle implementation of the SVGD algorithm illustrates the convergence of $\mathrm{KSD}^2(\mu_l^n|\pi)$ and $\mathrm{KL}(k \star \mu_l^n|\pi)$ to 0.

Open question 2: SVGD quantisation

The quality of a set of points $(x^1, \ldots, x^n)$ can be measured by the integral approximation error:

$$E(x_1, \ldots, x_n) = \left| \frac{1}{n} \sum_{i=1}^{n} f(x^i) - \int_{\mathbb{R}^d} f(x) d\pi(x) \right|. \tag{1}$$



(a) i.i.d. (b) SVGD Gaussian k (c) SVGD Laplace k

For i.i.d. points or MCMC iterates, (1) is of order $n^{-\frac{1}{2}}$. Can we bound (1) for SVGD final states?

Accurate quantization of measures via interacting particle-based optimization. Xu, L., Korba, A., Slepčev, D. ICML 2022.

Outline

A lot of problems previously came from the fact that the KL is not defined for discrete measures μ_n. Can we consider functionals that are well-defined for μ_n?

A lot of problems previously came from the fact that the KL is not defined for discrete measures μ_n. Can we consider functionals that are well-defined for μ_n?

Remember the Kernel Stein discrepancy of μ relative to π:

$$\mathrm{KSD}^2(\mu|\pi) = \left\| P_{\mu,k} \nabla \log\left(\frac{\mu}{\pi}\right) \right\|^2_{\mathcal{H}_k}, \; P_{\mu,k} : f \mapsto \int f(x) k(x,.) d\mu(x).$$

With several integration by parts we have:

$$\begin{aligned}
\mathrm{KSD}^2(\mu|\pi) &= \left\| P_{\mu,k} \nabla \log\left(\frac{\mu}{\pi}\right) \right\|^2_{\mathcal{H}_k} \\
&= \int\int \nabla \log\left(\frac{\mu}{\pi}(x)\right) \nabla \log\left(\frac{\mu}{\pi}(y)\right) k(x,y) d\mu(x) d\mu(y) \\
&= \iint \nabla \log \pi(x)^T \nabla \log \pi(y) k(x,y) + \nabla \log \pi(x)^T \nabla_2 k(x,y) \\
&\quad + \nabla_1 k(x,y)^T \nabla \log \pi(y) + \boldsymbol{\nabla} \cdot_1 \nabla_2 k(x,y) d\mu(x) d\mu(y) \\
&:= \iint k_\pi(x,y) d\mu(x) d\mu(y).
\end{aligned}$$

can be written in closed-form for discrete measures μ.

KSD Descent - algorithms [Korba et al., 2021]

We propose two ways to implement KSD Descent:

Algorithm 1 KSD Descent GD

Input: initial particles $(x_0^i)_{i=1}^N \sim \mu_0$, number of iterations M, step-size γ

for $n = 1$ **to** M **do**

$$[x_{n+1}^i]_{i=1}^N = [x_n^i]_{i=1}^N - \frac{2\gamma}{N^2} \sum_{j=1}^N [\nabla_2 k_\pi(x_n^j, x_n^i)]_{i=1}^N,$$

end for

Return: $[x_M^i]_{i=1}^N$.

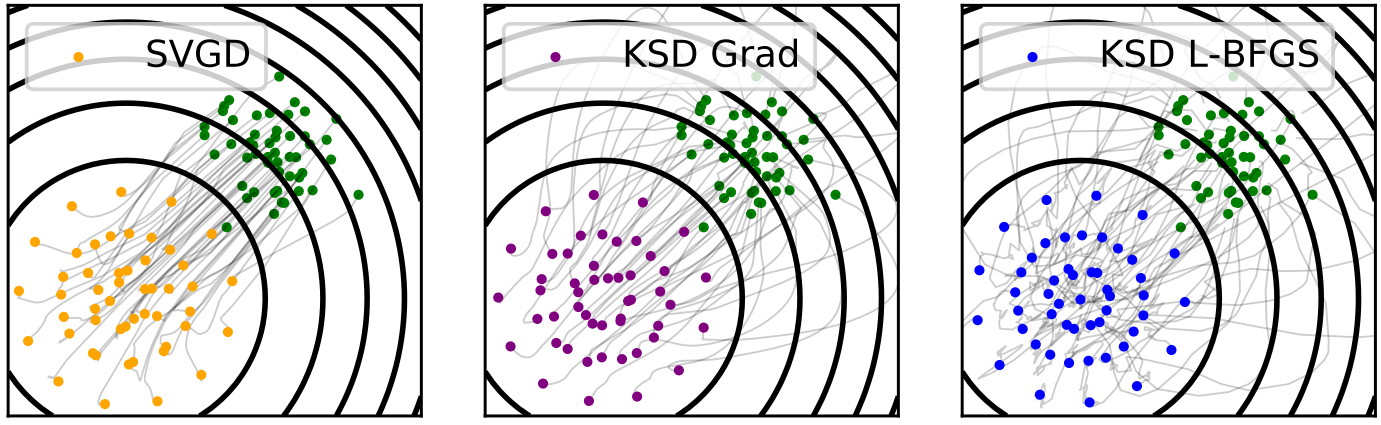
Algorithm 2 KSD Descent L-BFGS

Input: initial particles $(x_0^i)_{i=1}^N \sim \mu_0$, tolerance tol

Return: $[x_*^i]_{i=1}^N = \text{L-BFGS}(L, \nabla L, [x_0^i]_{i=1}^N, \text{tol})$.

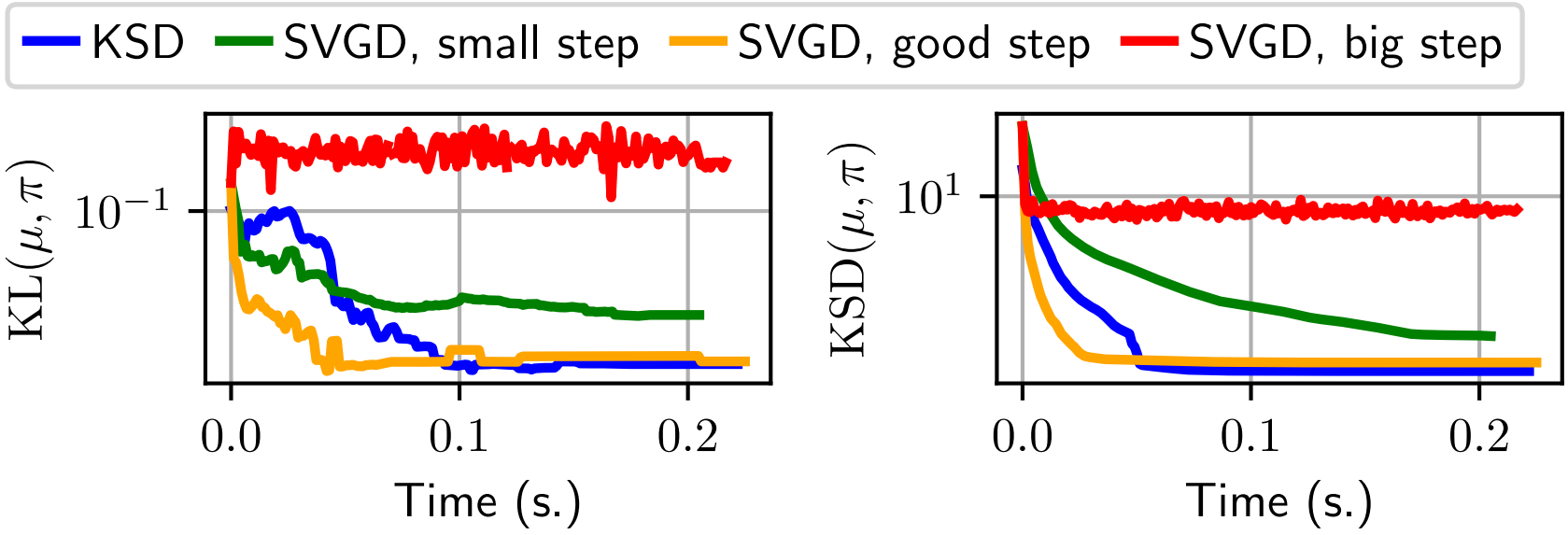
L-BFGS [Liu and Nocedal, 1989] is a quasi Newton algorithm that is faster and more robust than Gradient Descent, and **does not require the choice of step-size!**

Toy experiments - 2D standard gaussian



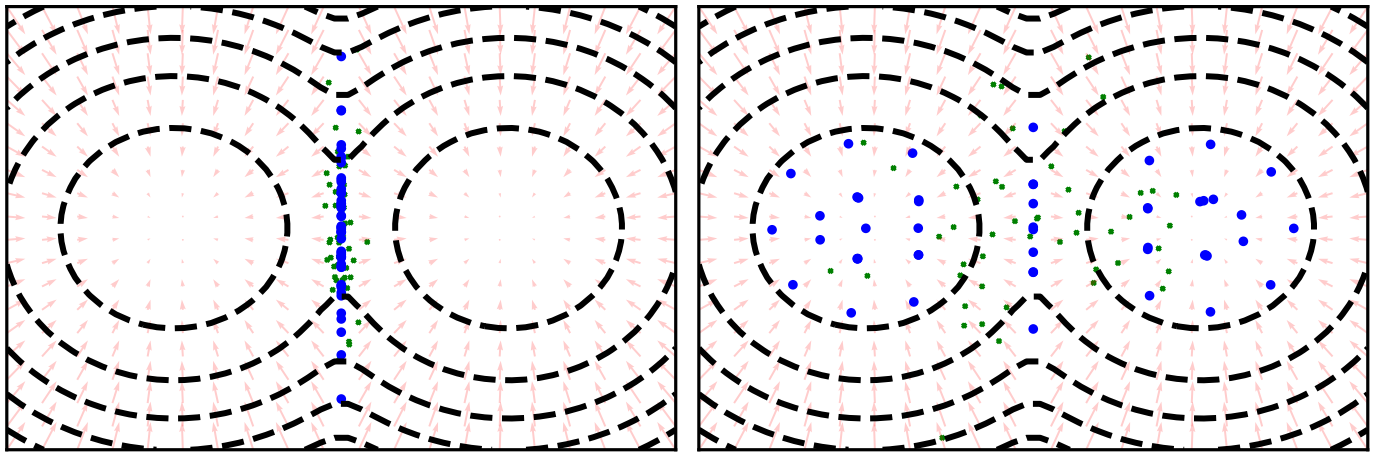
The green points represent the initial positions of the particles.
The light grey curves correspond to their trajectories.

SVGD vs KSD Descent - importance of the step-size



Convergence speed of KSD and SVGD on a Gaussian problem in 1D, with 30 particles.

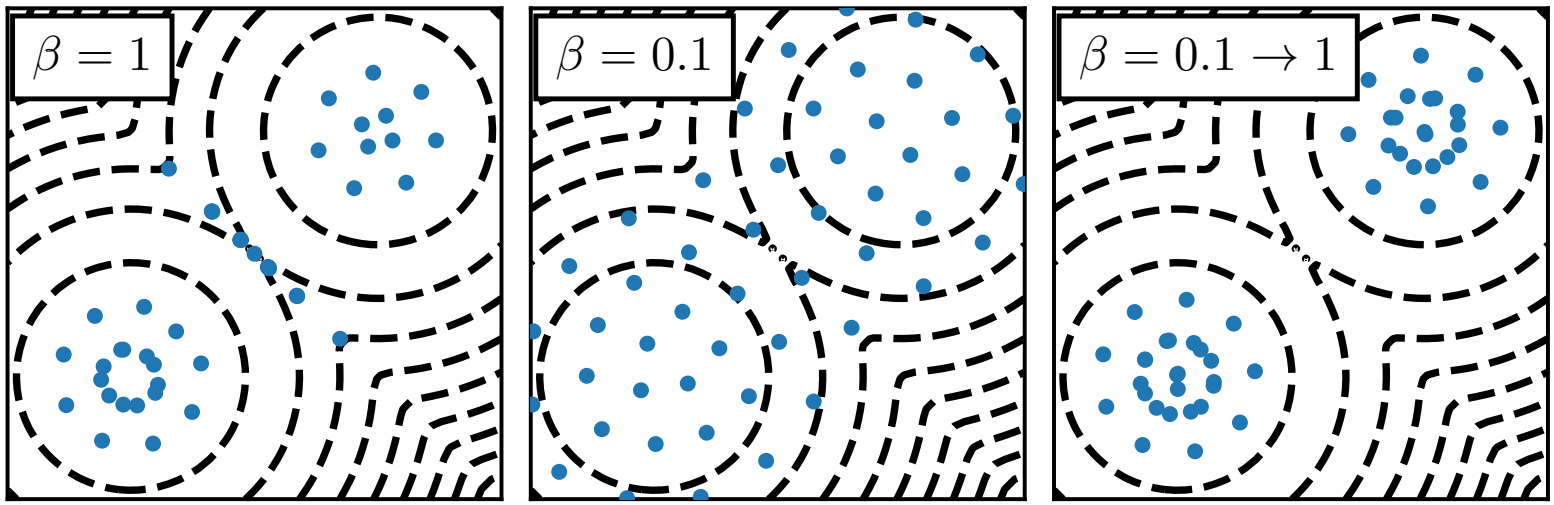
2D mixture of (isolated) Gaussians - failure cases



The green crosses indicate the initial particle positions
the blue ones are the final positions
The light red arrows correspond to the score directions.

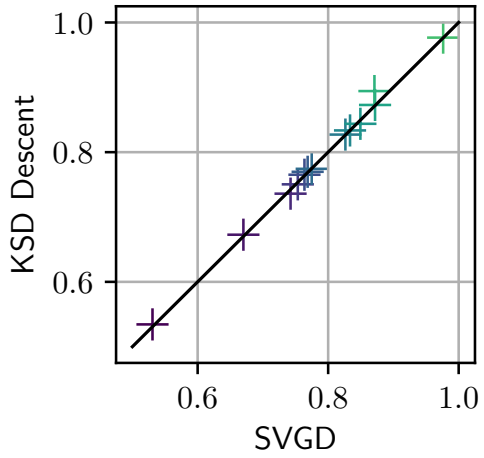
Isolated Gaussian mixture - annealing

Add an inverse temperature variable β : $\pi^{\beta}(x) \propto \exp(-\beta V(x))$, with $0 < \beta \leq 1$ (i.e. multiply the score by β.)

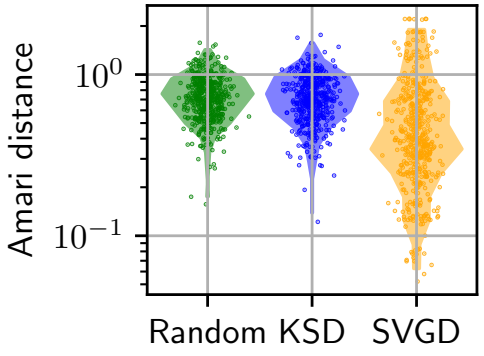


This is a hard problem, even for Langevin diffusions, where tempering strategies also have been proposed [Lee et al., 2018].

Real world experiments (10 particles)

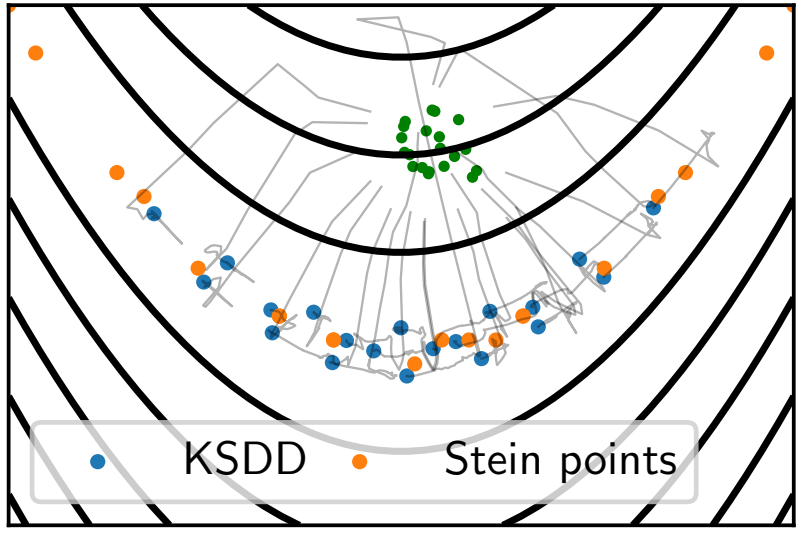


Bayesian logistic regression.
Accuracy of the KSD descent and SVGD for 13 datasets ($d \approx 50$).
Both methods yield similar results. KSD is better by 2% on one dataset.
Hint: convex likelihood.



Bayesian ICA.
Each dot is the Amari distance between an estimated matrix and the true unmixing matrix ($d \leq 8$).
KSD is not better than random.
Hint: highly non-convex likelihood.

So.. when does it work?



Comparison of KSD Descent and Stein points on a “banana” distribution. Green points are the initial points for KSD Descent. Both methods work successfully here, **even though it is not a log-concave distribution.**

We posit that KSD Descent succeeds because **there is no saddle point in the potential.**

Theoretical properties of KSD flow

Stationary measures:

- we show that if a stationary measure μ_∞ is full support, then $\mathcal{F}(\mu_\infty) = 0$.
- however, we also show that if *supp*$(\mu_0) \subset \mathcal{M}$, where $\mathcal{M}$ is a plane of symmetry of π, then for any time t it remains true for μ_t: *supp*$(\mu_t) \subset \mathcal{M}$.

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Explain convergence in the log-concave case? again an open question:

- the KSD is not geodesically convex
- it is not strongly geo convex near the global optimum π
- convergence of the continuous dynamics can be shown with a functional inequality, but which does not hold for discrete measures

KSD quantization

Theorem (Xu, K., Slečev): Assume that k is a Gaussian kernel and that $\pi \propto \exp(-U)$ where $U \in C^\infty(\mathbb{R}^d)$ is such that $U(x) > c_1|x|$ for large enough x, there exists polynomial f with degree m such that $\|\partial^\alpha U(x)\| \leq f(x)$ for all $1 \leq |\alpha| \leq d$. Then there exist points $x_1, ..., x_n$ such that $\mu_n = \sum_{i=1}^n \delta_{x_i}$ satisfies:

$$\mathrm{KSD}(\mu_n|\pi) \leq C_d \frac{(\log n)^{\frac{6d+2m+1}{2}}}{n}.$$

Note that for Gaussian mixtures π satisfies the conditions of the theorem.

Conclusion

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- They can provide a better approximation of the target for a finite number of particles
- Theory does not match practice yet
- Numerics can be improved, via perturbed dynamics, change of geometry...

Python package to try KSD descent and SVGD:
pip install ksddescent
website: `pierreablin.github.io/ksddescent/`

```
>>> import torch
>>> from ksddescent import ksdd_lbfgs
>>> n, p = 50, 2
>>> x0 = torch.rand(n, p)  # start from uniform distribution
>>> score = lambda x: x  # simple score function
>>> x = ksdd_lbfgs(x0, score)  # run the algorithm
```

Thank you!

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