Signal Reconstruction using Determinantal Sampling

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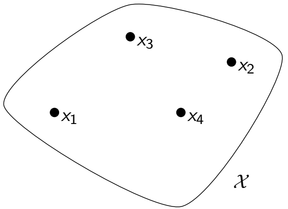
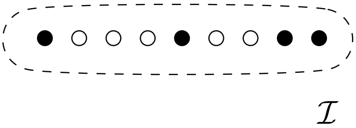
LJLL, Sorbonne Université - Paris

A **determinantal point process** (DPP) is a **distribution over subsets** of some set $\mathcal{X}, \mathcal{I}, \ldots$





Introduction

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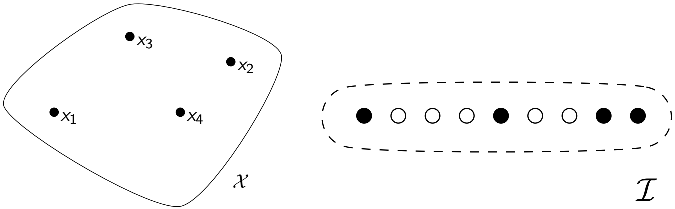
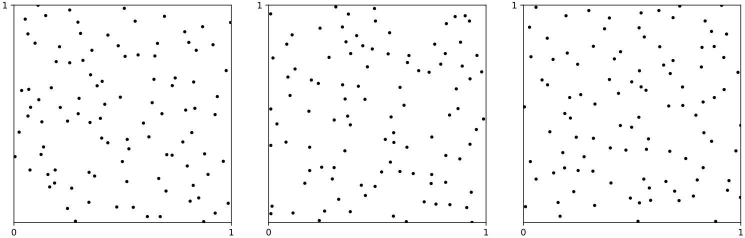


...with the **negative correlation** property:

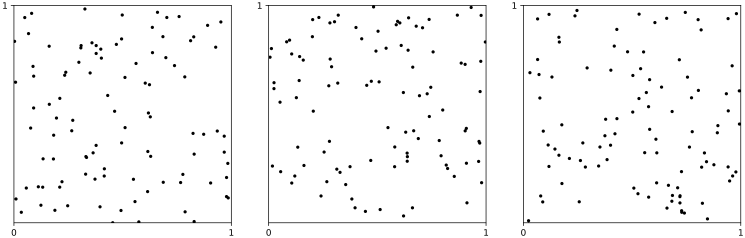
$$\forall B, B' \subset \mathcal{X}, \;\; B \cap B' = \emptyset \implies \mathbb{C}\mathrm{ov}(n_{\boldsymbol{x}}(B), n_{\boldsymbol{x}}(B')) \leq 0,$$

$$\text{where } n_{\boldsymbol{x}}(B) := |B \cap \boldsymbol{x}|$$

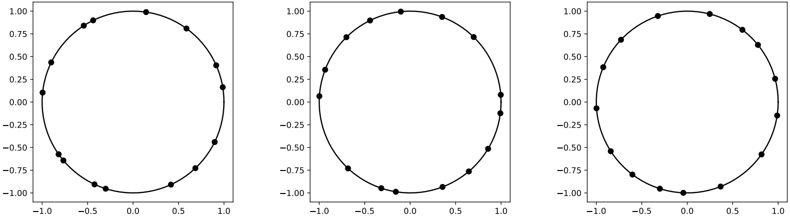
A DPP on $[0,1]^2$



i.i.d. particles on $[0,1]^2$



A DPP on the unit circle



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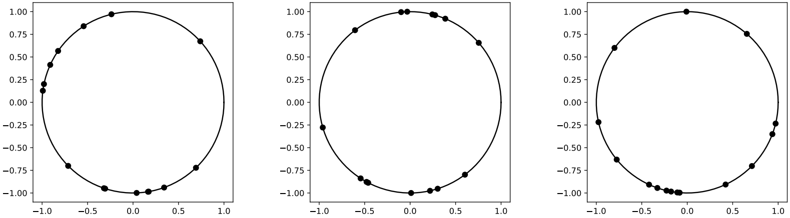

Introduction

- Early appearances of DPPs may be traced back to the work of Dyson (1962) [1] and Ginibre (1965)[2]
- A $\underbrace{\textbf{universal}}_{\text{generic } (\mathcal{X}, \omega)}$ definition is given in the work of Macchi (1975)[3]

[1] Dyson, F.J., 1962. Statistical theory of the energy levels of complex systems. I. Journal of Mathematical Physics, 3(1), pp.140-156.

[2] Ginibre, J., 1965. Statistical ensembles of complex, quaternion, and real matrices. Journal of Mathematical Physics, 6(3), pp.440-449.

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Definition (informal): Determinantal Point Process

Given a **metric space** $\mathcal{X}$ and a **measure** ω, a DPP satisfies

$$\mathbb{P}_{\mathrm{DPP}}\left(\substack{\exists \text{ at least } k \text{ points} \\ \text{one in each } B_i,\, i=1,\dots,k}\right) = \int_{B_1\times\cdots\times B_k} \mathrm{Det}\, \underbrace{\boldsymbol{\kappa}(x_1,\dots,x_k)}_{\text{kernel matrix}} \mathrm{d}\omega(x_1)\dots\mathrm{d}\omega(x_k)$$

for a **kernel** $\kappa : \mathcal{X}\times\mathcal{X}\to\mathbb{R}$.

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Introduction

Sampling using DPPs until 2019...

Discrete	Continuous
learning on budget [4]	numerical integration
node selection in a graph [5]	$\mathcal{X} \subset \mathbb{R}$ [6]
$\underbrace{\text{feature selection}}_{\text{universal results}}$ [7]	$\underbrace{\mathcal{X} = [0,1]^d}_{\text{specific results}}$ [8]

[4][Deshpande et al. (2006)],[Derezinski et al. (2017,2018,2019)]
[5][Tremblay et al. (2017)]
[6][Lambert (2018)]
[7][Belhadji et al. (2018)]
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universal results for
DPP-based sampling in continuous domain?

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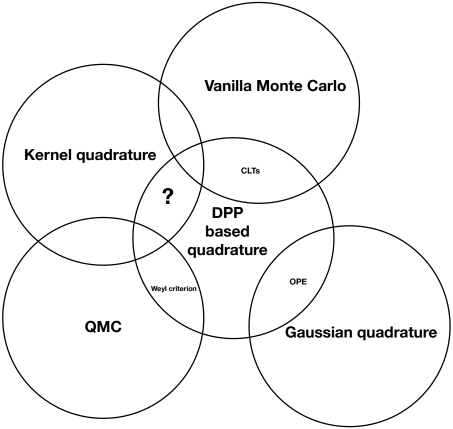
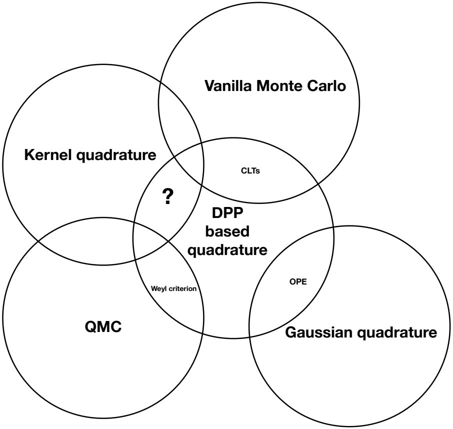
Outline

Towards a universal construction of quadrature rules



Towards a universal construction of quadrature rules

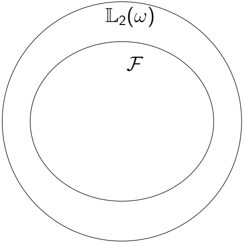


A universal construction of quadrature rules using DPPs?

Kernel-based analysis of quadrature rules

The crux of **kernel-based analysis** of a quadrature rule is the study of the **worst case error on the unit ball** of a RKHS $\mathcal{F}$

$$\sup_{\|f\|_{\mathcal{F}} \leq 1} \Big| \underbrace{\int_{\mathcal{X}} f(x) g(x) \mathrm{d}\omega(x)}_{\text{integral}} - \underbrace{\sum_{i \in [N]} w_i f(x_i)}_{\text{quadrature rule}} \Big|$$



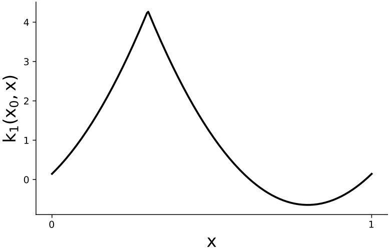
Reproducing kernel Hilbert spaces

Definition

An RKHS $\mathcal{F}$ is a Hilbert space associated to a kernel k, satisfying:

- $\forall x \in \mathcal{X},\ f \mapsto f(x)$ is continuous
- $\forall (x, f) \in \mathcal{X} \times \mathcal{F},\ \langle f, k(x, .)\rangle_{\mathcal{F}} = f(x)$

Example: $\mathcal{X} = [0, 1]$ and $k_1(x, y) := 1 - 2\pi^2 \mathcal{B}_2(\{x - y\})$ where $\{x - y\}$ is the fractional part of $x - y$, and $\mathcal{B}_2(x) = x^2 - x + \frac{1}{6}$.



Embeddings as elements of the RKHS

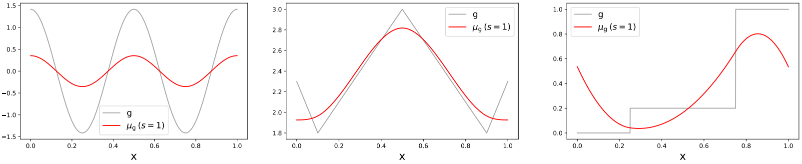
$\mathcal{F}$ contains smooth functions

Definition: an embedding of an element of $\mathbb{L}_2(\omega)$

Given $g \in \mathbb{L}_2(\omega)$, the *embedding* of g is defined by

$$\mu_g = \mathbf{\Sigma} g := \int_{\mathcal{X}} k(x,.)g(x)\mathrm{d}\omega(x) \in \mathcal{F}$$

$\mathbf{\Sigma} : \mathbb{L}_2(\omega) \to \mathbb{L}_2(\omega) =$ **integration operator** associated to (k, ω).



Embeddings and quadrature rules

Properties

- The **reproducibility of integrals**

$$\forall f \in \mathcal{F},\ \langle f, \mu_g \rangle_{\mathcal{F}} = \int_{\mathcal{X}} f(x) g(x) \mathrm{d}\omega(x).$$

- The **worst integration error on the unit ball** of $\mathcal{F}$:

$$\sup_{\|f\|_{\mathcal{F}} \leq 1} \Big| \underbrace{\int_{\mathcal{X}} f(x) g(x) \mathrm{d}\omega(x)}_{\text{integral}} - \underbrace{\sum_{i=1}^{N} w_i f(x_i)}_{\text{quadrature rule}} \Big| = \underbrace{\Big\| \mu_g - \sum_{i=1}^{N} w_i k(x_i, .) \Big\|_{\mathcal{F}}}_{\text{WCE}}$$

The study of quadrature rules boils down to the study of kernel approximations of embeddings

Embeddings and quadrature rules

A sanity check using the 'Monte Carlo quadrature'

Let $x_1, \dots, x_N = \text{i.i.d.} \sim \omega, w_i = 1/N$.
Under some assumptions on k, we have

$$\mathbb{E}\Big\|\mu_g - \sum_{i=1}^{N} \frac{1}{N} k(x_i, .)\Big\|_{\mathcal{F}}^2 = \mathcal{O}(1/N).$$

We recover the 'Monte Carlo rate' $\mathcal{O}(1/N)$

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Can we do better?

The optimal kernel quadrature

Definition: the optimal kernel quadrature

Given a set of nodes $\boldsymbol{x} = \{x_1, \dots, x_N\}$ s.t.

$$\boldsymbol{K}(\boldsymbol{x}) := \begin{pmatrix} k(x_1, x_1) & \dots & k(x_1, x_N) \\ \vdots & \ddots & \vdots \\ k(x_N, x_1) & \dots & k(x_N, x_N) \end{pmatrix}$$

is non-singular, the **optimal kernel quadrature** is the couple $(\boldsymbol{x}, \hat{\boldsymbol{w}})$ such that

$$\left\| \mu_g - \sum_{i=1}^{N} \hat{w}_i(\mu_g) k(x_i, .) \right\|_{\mathcal{F}} = \min_{\boldsymbol{w} \in \mathbb{R}^N} \left\| \mu_g - \sum_{i=1}^{N} w_i k(x_i, .) \right\|_{\mathcal{F}}$$

The convergence of the optimal kernel quadrature

The study of the **convergence rate** of the optimal kernel quadrature was carried out in several works

$\mathcal{X}$	$\mathcal{F}$ or k	$\boldsymbol{x}$	The rate	Reference
$[0,1]$	Sobolev S.	Unif. grid (g is cos or sin)	$\mathcal{O}(N^{-2s})$	[Novak et al., 2015] [Bojanov , 1981]
$[0,1]^d$	$\otimes$ Sobolev S.	QMC seq. (g is constant)	QMC rates	[Briol et al, 2019]
$[0,1]^d$	$\otimes$ Sobolev S.	-	$\mathcal{O}(N^{-2s/d})$	[Wendland, 2004]
$\mathbb{R}^d$	Gaussian	$\otimes$ Hermite roots (+ assumptions)	$\mathcal{O}(\exp(-\alpha N))$	[Karvonen et al., 2019]
...	...	...	...	...

Limitation

The analysis is **too specific** to the RKHS $\mathcal{F}$, to $\boldsymbol{x}$, to g...

Any universal analysis of the OKQ?

A spectral characterization of the RKHS

What is common among existing results?

Assumptions

- There exists a **spectral decomposition** $(e_m, \sigma_m)_{m \in \mathbb{N}^*}$ of $\boldsymbol{\Sigma}$ where $(e_m)_{m \in \mathbb{N}^*}$ is an o.n.b. of $\mathbb{L}_2(\omega)$ and $\sigma_1 \geq \sigma_2 \geq ... > 0$ s.t.

$$\sum_{m \in \mathbb{N}^*} \sigma_m < +\infty$$

- The RKHS $\mathcal{F}$ is **dense** in $\mathbb{L}_2(\omega)$

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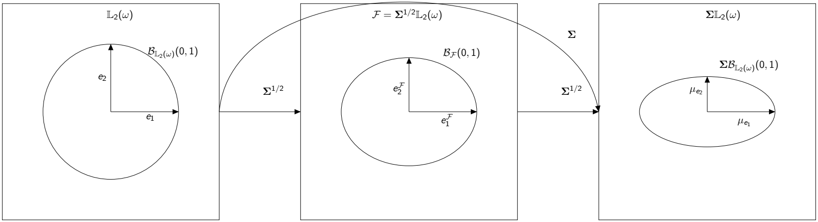
- The RKHS $\mathcal{F}$ is **dense** in $\mathbb{L}_2(\omega)$

$$\|f\|_{\mathcal{F}}^2 = \sum_{m \in \mathbb{N}^*} \frac{\langle f, e_m \rangle_\omega^2}{\sigma_m} \implies \{\|f\|_{\mathcal{F}} = 1\} \textbf{ is an ellipsoid in } \mathbb{L}_2(\omega)$$

$$\sum_{m \in \mathbb{N}^*} \frac{\langle \mu_g, e_m \rangle_\omega^2}{\sigma_m^2} = \sum_{m \in \mathbb{N}^*} \langle g, e_m \rangle_\omega^2 \implies \{\mu_g; \|g\|_\omega = 1\} \textbf{ is an ellipsoid in } \mathbb{L}_2(\omega)$$

A spectral characterization of the RKHS



$\mathcal{X}$	$\mathcal{F}$ or k	σ_{N+1}	(e_m)
$[0,1]$	Sobolev	$\mathcal{O}(N^{-2s})$	Fourier
$[0,1]^d$	Korobov	$\mathcal{O}(\log(N)^{2s(d-1)}N^{-2s})$	$\otimes$ of Fourier
$[0,1]^d$	Sobolev	$\mathcal{O}(N^{-2s/d})$	"Fourier"
$\mathbb{S}^d$	Dot product	"-"	Spherical Harmonics
$\mathbb{R}$	Gaussian	$\mathcal{O}(e^{-\alpha N})$	Hermite Polys.
$\mathbb{R}^d$	Gaussian	$\mathcal{O}(e^{-\alpha d N^{1/d}})$	$\otimes$ of Hermite Polys.
...	...	...	...

The determinantal distributions

Definition

Let κ be a kernel s.t. $\int_{\mathcal{X}} \kappa(x,x)\mathrm{d}\omega(x) < +\infty$. The function

$$p_{\kappa}(x_1, \dots, x_N) \propto \operatorname{Det} \boldsymbol{\kappa}(\boldsymbol{x}) = \operatorname{Det} \begin{pmatrix} \kappa(x_1, x_1) & \dots & \kappa(x_1, x_N) \\ \vdots & \ddots & \vdots \\ \kappa(x_N, x_1) & \dots & \kappa(x_N, x_N) \end{pmatrix}$$

is a p.d.f. on $\mathcal{X}^N$ w.r.t. $\omega^{\otimes N}$.

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Theorem (Belhadji et al. (2020))

For $(x_1,\dots,x_N) \sim p_\kappa$, the set $\boldsymbol{x} := \{x_1,\dots,x_N\}$ follows the distribution of a **mixture of DPPs**.

Main results

Theorem (Belhadji, Bardenet and Chainais (2019))

Under the determinantal distribution corresponding to

$$\kappa(x,y) = \mathfrak{K}(x,y) := \sum_{n \in [N]} e_n(x) e_n(y),$$

$\boldsymbol{x}$ follows the distribution of a **(projection) DPP**, and we have

$$\mathbb{E}_{p_\kappa} \sup_{\|g\|_\omega \leq 1} \left\| \mu_g - \sum_{i=1}^{N} \hat{w}_i(\mu_g) k(x_i, .) \right\|_{\mathcal{F}}^2 \leq 4N^2 r_{N+1},$$

where $r_{N+1} = \sum_{m=N+1}^{+\infty} \sigma_m$.

σ_m	Theoretical rate ($N^2 r_{N+1}$)	Empirical rate
m^{-2s}	$N^3 \mathcal{O}(\sigma_{N+1})$	$\mathcal{O}(\sigma_{N+1})$
α^m	$N^2 \mathcal{O}(\sigma_{N+1})$ $\approx \mathcal{O}(\sigma_{N+1})$	$\mathcal{O}(\sigma_{N+1})$

Main results

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$$\forall g \in \mathbb{L}_2(\omega),\ \ \mathbb{E}_{p_\kappa} \left\| \mu_g - \sum_{i=1}^{N} \hat{w}_i(\mu_g) k(x_i, .) \right\|_{\mathcal{F}}^2 \leq 4 \|g\|_\omega^2 r_{N+1}$$

$$= \mathcal{O}(r_{N+1})$$

where $r_{N+1} := \sum_{m=N+1}^{+\infty} \sigma_m$.

Main results

Theorem (Belhadji, Bardenet and Chainais (2020))

Under the determinantal distribution corresponding to $\kappa = k$, we have

$$\forall g \in \mathbb{L}_2(\omega), \;\; \mathbb{E}_{p_\kappa} \left\| \mu_g - \sum_{i=1}^{N} \hat{w}_i(\mu_g) k(x_i, .) \right\|_{\mathcal{F}}^2 = \sum_{m=1}^{+\infty} \langle g, e_m \rangle_\omega^2 \epsilon_m(N)$$
$$\leq \|g\|_\omega^2 \epsilon_1(N)$$

where

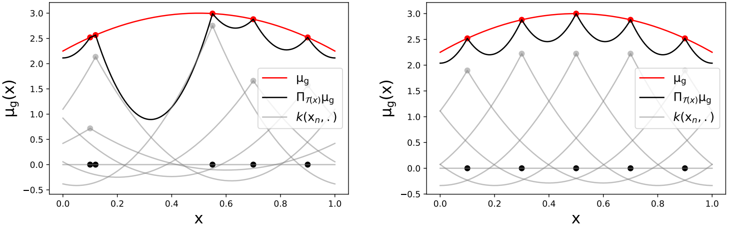
$$\epsilon_m(N) = \sum_{\substack{T \subset \mathbb{N}^* \smallsetminus \{m\} \\ |T| := N}} \prod_{t \in T} \sigma_t \Bigg/ \sum_{\substack{T \subset \mathbb{N}^* \\ |T| = N}} \prod_{t \in T} \sigma_t = \underbrace{\mathcal{O}(\sigma_{N+1})}_{\substack{\text{optimal rate} \\ \text{(Pinkus (1985))}}}$$

The optimal kernel quadrature and kernel interpolation

The mixture corresponding to OKQ is an interpolant

The function $\hat{\mu}_g := \sum_{i=1}^{N} \hat{w}_i(\mu_g)k(x_i, .)$ satisfies

$$\forall i \in [N], \ \hat{\mu}_g(x_i) = \mu_g(x_i).$$



Main results

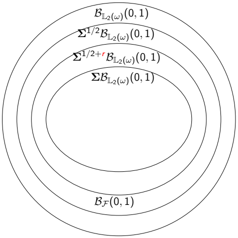
Theorem (Belhadji, Bardenet and Chainais (2020))

Under the determinantal distribution corresponding to $\kappa = k$, for $r \in [0, 1/2]$, we have

$$\forall f \in \mathbf{\Sigma}^{1/2+r}\mathbb{L}_2(\omega), \;\; \mathbb{E}_{p_\kappa} \left\| f - \sum_{i=1}^{N} \hat{w}_i(f) k(x_i, .) \right\|_{\mathcal{F}}^2 = \mathcal{O}({\sigma_{N+1}}^{2r})$$

$r = 0 \rightarrow$ a generic element of $\mathcal{F}$

$r = 1/2 \rightarrow$ an embedding of some $g \in \mathbb{L}_2(\omega)$



Main results

The RKHS norm $\|.\|_\mathcal{F}$ is strong: $\|f\|_\infty \leq \sup_{x \in \mathcal{X}} k(x,x) \|f\|_\mathcal{F}$

We seek convergence guarantees in a **weaker norm** such as $\|.\|_\omega$

Main results

Definition: least squares approximation

Given $\boldsymbol{x} = \{x_1, \dots, x_N\} \subset \mathcal{X}$, the **least squares approximation** of $f \in \mathcal{F}$ associated to $\boldsymbol{x}$ is the function $\hat{f}_{\mathrm{LS},\boldsymbol{x}}$ defined by

$$\|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_\omega = \min_{\hat{f} \in \mathcal{T}(\boldsymbol{x})} \|f - \hat{f}\|_\omega,$$

where $\mathcal{T}(\boldsymbol{x}) := \mathrm{Span}(k(x_1, .), \dots, k(x_N, .))$.

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$$\neq$$

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$$\|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|_{\mathcal{F}} = \min_{\hat{f} \in \mathcal{T}(\boldsymbol{x})} \|f - \hat{f}\|_{\mathcal{F}}.$$

Main results

Theorem (Belhadji, Bardenet and Chainais (2023))

Under the determinantal distribution corresponding to $\kappa = \mathfrak{K}$, we have

$$\forall f \in \mathcal{F}, \; \mathbb{E}_{p_\kappa} \|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_\omega^2 \leq 2\Big(\underbrace{\|f - f_N\|_\omega^2}_{\mathcal{O}(\sigma_{N+1})} + \|f_N\|_\omega^2 \underbrace{\sum_{m=N+1}^{+\infty} \sigma_m^2}_{\substack{\mathcal{O}(\sigma_{N+1} r_{N+1}) \\ \textbf{superconvergence}}} \Big)$$

where $r_{N+1} = \sum_{m=N+1}^{+\infty} \sigma_m$.

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where $r_{N+1} = \sum_{m=N+1}^{+\infty} \sigma_m$.

The computation of $\hat{f}_{\mathrm{LS},\boldsymbol{x}}$ requires the values of $\mu_f(x_1), \dots, \mu_f(x_N)$ not $f(x_1), \dots, f(x_N)$.

The empirical least squares approximation

Definition (Cohen, Davenport and Leviatan (2013))

Let $\boldsymbol{x} \in \mathcal{X}^N$ and $q : \mathcal{X} \to \mathbb{R}_+^*$. Consider the so-called *empirical* semi-norm $\|.\|_{q,\boldsymbol{x}}$ defined on $\mathbb{L}_2(\omega)$ by

$$\|h\|_{q,\boldsymbol{x}}^2 := \frac{1}{N} \sum_{i=1}^{N} q(x_i) h(x_i)^2.$$

The empirical least squares approximation is defined as

$$\hat{f}_{\mathrm{ELS},M,\boldsymbol{x}} := \arg\min_{\hat{f} \in \mathcal{E}_M} \|f - \hat{f}\|_{q,\boldsymbol{x}}^2,$$

where $\mathcal{E}_M := \mathrm{Span}(e_1, \ldots, e_M)$.

When $M = N$, the function $\hat{f}_{\mathrm{ELS},M,\boldsymbol{x}}$ does not depend on q.

Theorem (Belhadji, Bardenet and Chainais (2023))

Consider $N, M \in \mathbb{N}^*$ such that $M \leq N$. Let $f \in \mathcal{F}$, and define

$$\hat{f}_{\mathrm{tELS},M,\boldsymbol{x}} := \sum_{m=1}^{M} \langle \hat{f}_{\mathrm{ELS},N,\boldsymbol{x}}, e_m \rangle_\omega e_m.$$

Under the determinantal distribution corresponding to $\kappa = \mathfrak{K}$ (of cardinality N), we have

$$\mathbb{E}_{p_\kappa} \|f - \hat{f}_{\mathrm{tELS},M,\boldsymbol{x}}\|_\omega^2 = \|f - f_M\|_\omega^2 + M\|f - f_N\|_\omega^2.$$

In particular,

$$\underbrace{\mathbb{E}_{p_\kappa} \|f - \hat{f}_{\mathrm{tELS},M,\boldsymbol{x}}\|_\omega^2 \leq (1+M)\|f - f_M\|_\omega^2}_{\text{'Instance Optimal Property' (IOP)}}.$$

Some remarks:

- $\mathbb{E}_{p_\kappa}\|f - \hat{f}_{\mathrm{tELS},N,\boldsymbol{x}}\|_\omega^2 \leq (1+N)\|f - f_N\|_\omega^2 = \mathcal{O}(N\sigma_{N+1})$
- In general $\hat{f}_{\mathrm{tELS},M,\boldsymbol{x}} \neq \hat{f}_{\mathrm{ELS},M,\boldsymbol{x}}$
- The IOP was proved for $\hat{f}_{\mathrm{ELS},M,\boldsymbol{x}}$ under Christoffel sampling[9]:

$$x_1, \dots, x_N \underbrace{\sim}_{\textbf{i.i.d.}} \frac{1}{M}\sum_{m=1}^{M} e_m(x)^2 \mathrm{d}\omega(x)$$

conditioned on the event $\{\|\boldsymbol{G} - \mathbb{I}\|_{\mathrm{op}} \geq 1/2\}$, where $\boldsymbol{G} := (\langle e_i, e_j\rangle_{q,\boldsymbol{x}})_{i,j\in[M]}$ is the Gramian matrix of the family $(e_j)_{j\in[M]}$associated to the empirical scalar product $\langle .,.\rangle_{q,\boldsymbol{x}}$

[9]Cohen, A. and Migliorati, G., 2017. Optimal weighted least-squares methods. The SMAI journal of computational mathematics, 3, pp.181-203.

The determinantal distribution corresponding associated to the kernel $\kappa(x,y) := \frac{1}{M}\sum_{m=1}^{M} e_m(x)e_m(y)$ is a **natural extension of Christoffel sampling**

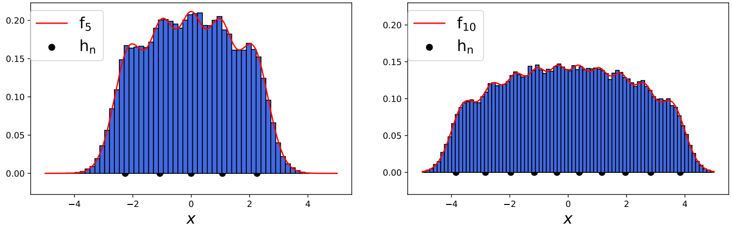
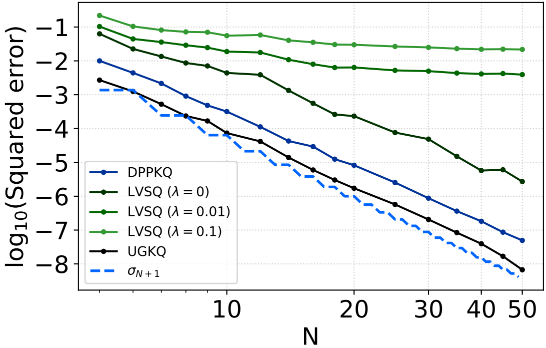


Figure: Histograms of 50000 realizations of the projection DPP associated to the first N Hermite polynomials, compared to the inverse of Christoffel function, and the nodes of the Gaussian quadrature with $N \in \{5, 10\}$

We report the empirical expectation of a surrogate of the worst interpolation error

$$\mathbb{E}_{\kappa} \sup_{\substack{f=\mu_g \\ \|g\|_\omega \leq 1}} \|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|_{\mathcal{F}}^2 \approx \mathbb{E}_{\kappa} \sup_{\substack{f=\mu_g \\ g \in \mathcal{G}}} \|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|_{\mathcal{F}}^2 \; ; \kappa = \mathfrak{K}$$

where $\mathcal{G} \subset \{g, \|g\|_\omega \leq 1\}$ is a finite set $|\mathcal{G}| = 5000$.
$\mathcal{F}$ is the periodic Sobolev space of order $s = 3$.



Numerical simulations

$\mathcal{F}$ = Korobov space of order $s = 1$, $\mathcal{X} = [0,1]^2$

We report $\epsilon_m(N) = \mathbb{E}_{\mathfrak{K}} \|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|^2_{\mathcal{F}}; \kappa = \mathfrak{K}$ where $f = \mu_{e_m}$

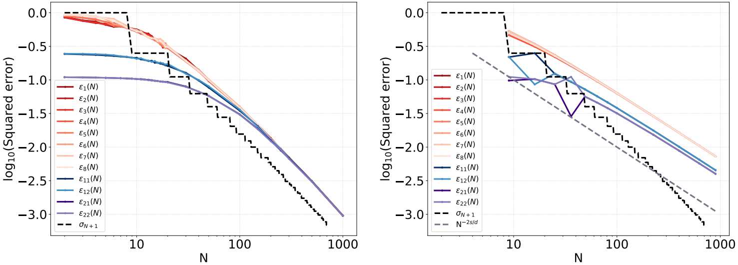


Figure: OKQ using DPPs (left) vs OKQ using the uniform grid (right)

Numerical simulations

$\mathcal{F}$ = the periodic Sobolev space of order $s = 1$, $\mathcal{X} = [0, 1]$

We report $\mathbb{E}_{\kappa}\|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_{\omega}^{2}$; $\kappa = \mathfrak{K}$ where

$$f = \sum_{m=1}^{M} \xi_m e_m^{\omega}; \quad \xi_1, \ldots, \xi_M \underbrace{\sim}_{\textbf{i.i.d}} \mathcal{N}(0,1)$$

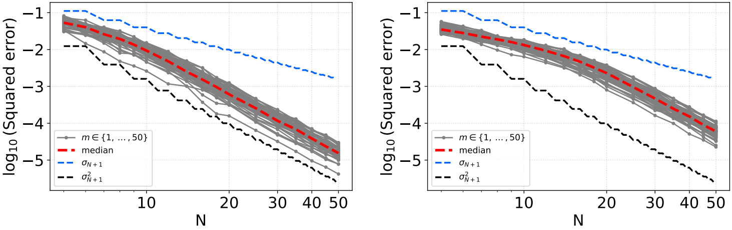


Figure: $M = 10$ (left) and $M = 20$ (right)

Numerical simulations

Consider $\mathcal{F}$ to be the RKHS defined by the Sinc kernel

$$k(x,y) = \frac{\sin(F(x-y))}{F(x-y)};\ \ \mathcal{X} = [-T/2, T/2]$$

- The eigenfunctions e_m correspond to the prolate spheroidal wave functions[10] (PSWF)
- The asymptotics of the eigenvalues in the limit $c := TF \to +\infty$ were investigated[11]: there is approximately c eigenvalues close to 1, and the remaining eigenvalues decrease to 0 at an exponential rate.

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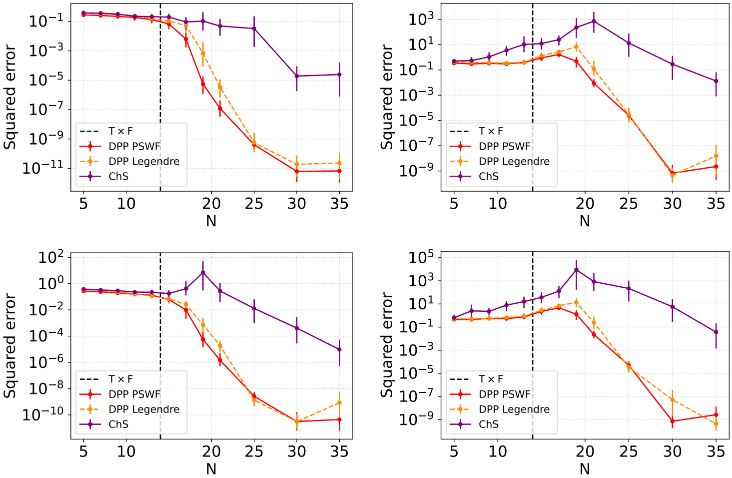
[10] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, fourier analysis and uncertainty— i. Bell System Technical Journal, 40(1):43 63, 1961.

[11] H. J. Landau and H. Widom. Eigenvalue distribution of time and frequency limiting. Journal of Mathematical Analysis and Applications, 77(2):469 481, 1980.

Numerical simulations

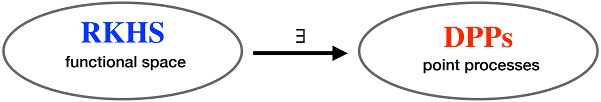
We report $\|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_\omega^2$ averaged over 50 realizations for $f \in \{e_1, e_2, e_3, e_4\}$.



Conclusion

Take Home Messages

- The theoretical study of the **optimal kernel quadrature** under **determinantal sampling**
- The study of **function reconstruction** under **determinantal sampling**
- The analysis is **universal**
- **Empirical validation** on various RKHSs
- Projection DPPs are natural extensions of Christoffel sampling, and yield better empirical results



Perspectives: extending the theoretical results?

The study of

- $\mathbb{E}\|f - \hat{f}_{\mathrm{ELS},M,\boldsymbol{x}}\|_\omega^2$ when dimension $M \neq N$ number of nodes
- $\mathbb{E}\sup_{\|f\|_{\mathcal{F}}\leq 1} \|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_\omega^2$ instead of $\mathbb{E}\|f - \hat{f}_{\mathrm{LS},\boldsymbol{x}}\|_\omega^2$?
- $\mathbb{E}\sup_{\|f\|_{\mathcal{F}}\leq 1} \|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|_\omega^2$ and $\mathbb{E}\|f - \hat{f}_{\mathrm{OKA},\boldsymbol{x}}\|_\omega^2$?
- High order moments?
- ...

Perspectives: efficient sampling in continuous domain?

Let $\boldsymbol{x} = \{x_1, \dots, x_N\}$ such that $\operatorname{Det} \boldsymbol{\kappa}(\boldsymbol{x}) > 0$. We have

$$\begin{aligned}
\operatorname{Det} \boldsymbol{\kappa}(\boldsymbol{x}) = & \kappa(x_1, x_1) \\
& \times \left(\kappa(x_2, x_2) - \frac{\kappa(x_1, x_2)^2}{\kappa(x_1, x_1)} \right) \\
& \dots \\
& \times \left(\kappa(x_\ell, x_\ell) - \phi_{\boldsymbol{x}_{\ell-1}}(x_\ell)^T \boldsymbol{\kappa}(\boldsymbol{x}_{\ell-1})^{-1} \phi_{\boldsymbol{x}_{\ell-1}}(x_\ell) \right) \\
& \dots, \\
& \times \left(\kappa(x_N, x_N) - \phi_{\boldsymbol{x}_{N-1}}(x_N)^T \boldsymbol{\kappa}(\boldsymbol{x}_{N-1})^{-1} \phi_{\boldsymbol{x}_{N-1}}(x_N) \right)
\end{aligned}$$

where $\phi_{\boldsymbol{x}_{\ell-1}}(x) = (\kappa(\xi, x))^T_{\xi \in \boldsymbol{x}_{\ell-1}} \in \mathbb{R}^{\ell-1}$, $\boldsymbol{x}_{\ell-1} = \{x_1, \dots, x_{\ell-1}\}$.

Perspectives: efficient sampling in continuous domain?

Define

$$p_{\kappa,1}(x) = \kappa(x, x),$$
$$p_{\kappa,\ell}(x) = \kappa(x, x) - \phi_{\boldsymbol{x}_{\ell-1}}(x)^{\mathsf{T}} \boldsymbol{\kappa}(\boldsymbol{x}_{\ell-1})^{-1} \phi_{\boldsymbol{x}_{\ell-1}}(x); \;\; \ell \geq 2$$

If κ is a projection kernel

$$\int_{\mathcal{X}} p_{\kappa,\ell}(x) \mathrm{d}\omega(x) = N - \ell + 1,$$

and

$$p_{\kappa}(\boldsymbol{x}) := \frac{1}{N!} \operatorname{Det} \boldsymbol{\kappa}(\boldsymbol{x}) = \prod_{\ell=1}^{N} \frac{1}{N - \ell + 1} p_{\kappa,\ell}(x)$$

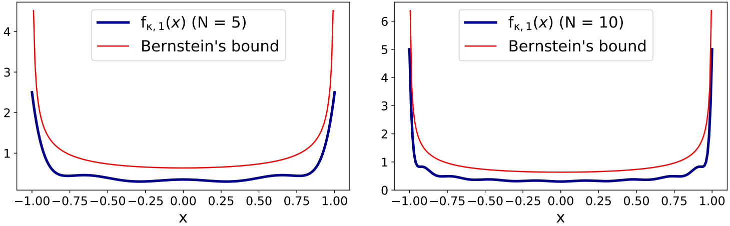
and the sequential algorithm is exact (the HKPV algorithm).

Perspectives: efficient sampling in continuous domain?

Example: to conduct Christoffel sampling for Legendre polynomials we can use the *Bernstein's bound*[12]

$$\forall n \in \mathbb{N}^*, \;\; L_n(x)^2 \leq \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}} \implies p_{\kappa,1}(x) \leq \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}},$$

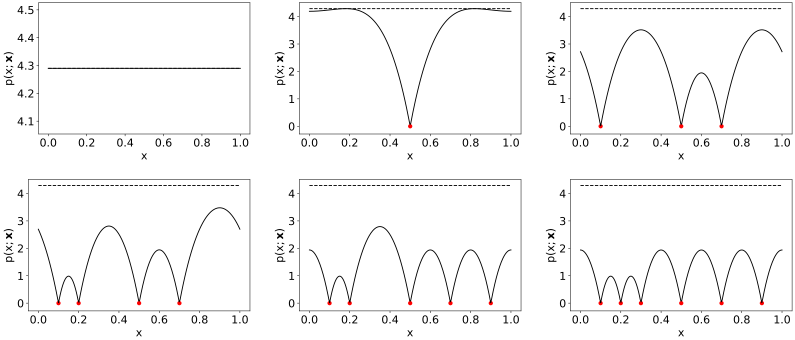
and use rejection sampling (proposal = Beta distribution).



[12]Lorch, L. (1983). "Alternative proof of a sharpened form of Bernstein's inequality for Legendre polynomials". In: Applicable Analysis 14.3, pp. 237–240.

Perspectives: efficient sampling in continuous domain?

One sample from a projection DPP (of cardinality N) requires to draw in average N^2 samples from the Christoffel distribution as a proposal: **sampling is getting harder as N get bigger**



Perspectives: efficient sampling in continuous domain?

Sampling from a projection DPP = looking for the eigenfunctions
+ looking for an upper bound for the inverse Christoffel function
+ looking for a sampling algorithm from the upper bound

How to address these issues in practice?

[13] Dolbeault, M. and Cohen, A., 2022. Optimal sampling and Christoffel functions on general domains. Constructive Approximation, 56(1), pp.121-163.

[14] Belhadji, A., Bardenet, R. and Chainais, P., 2020, November. Kernel interpolation with continuous volume sampling. In International Conference on Machine Learning (pp. 725-735). PMLR.

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How to address these issues in practice?

We may

- Look for upper bounds or asymptotics of the inverse of Christoffel function in some RKHS on general domains [13]

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Perspectives: efficient sampling in continuous domain?

Sampling from a projection DPP = looking for the eigenfunctions
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How to address these issues in practice?

We may

- Look for upper bounds or asymptotics of the inverse of Christoffel function in some RKHS on general domains [13]
- Work with continuous volume sampling → no need to spectral decomposition[14]

[13] Dolbeault, M. and Cohen, A., 2022. Optimal sampling and Christoffel functions on general domains. Constructive Approximation, 56(1), pp.121-163.

[14] Belhadji, A., Bardenet, R. and Chainais, P., 2020, November. Kernel interpolation with continuous volume sampling. In International Conference on Machine Learning (pp. 725-735). PMLR.

Perspectives: extension to atomic measure reconstruction?

Given a class of objects $\mathcal{M}$, is it possible to approximate the elements of $\mathcal{M}$ using its evaluation on some functionals:

$$L_1(\mu), \dots, L_N(\mu) \longrightarrow \hat{\mu} \approx \mu; \;\; \mu \in \mathcal{M}$$

The objects	Functions	Atomic measures
The class $\mathcal{M}$	RKHS	not a Hilbert space
Functionals $L_1, \dots, L_N$	$\mu \mapsto \mu(x_j)$	$\mu \mapsto \int e^{\mathbf{i}\omega_j^{\mathrm{T}} x} \mathrm{d}\mu(x)$
Distance preserving P.	IOP	RIP [15]
Decoding	$\hat{f}_{\mathrm{LS},\boldsymbol{x}}, \hat{f}_{\mathrm{OKA},\boldsymbol{x}}, \dots$	CL-OMP, Mean-shift [16]

[15] Belhadji, A. and Gribonval, R., 2022. Revisiting RIP guarantees for sketching operators on mixture models.

[16] Belhadji, A. and Gribonval, R., 2022. Sketch and shift: a robust decoder for compressive clustering

Thank you for your attention!

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ICML, 2020.

A. Belhadji
An analysis of Ermakov-Zolotukhin quadrature using kernels.
NeurIPS, 2021.

A. Belhadji, R. Bardenet, and P. Chainais
Signal reconstruction using determinantal sampling
arXiv:2310.09437.